Proofs, intuitions and diagrams

Rood, P.T.M.

2005

document version
Publisher's PDF, also known as Version of record

Proofs, intuitions and diagrams

Kant and the mathematical method of proof

Cover design: Edith Kuyvenhoven
(www.edith-made-it.nl)

Printed by Wöhrmann Print Service, Zutphen
(www.wps.nl)

VRIJE UNIVERSITEIT

Proofs, intuitions and diagrams

Kant and the mathematical method of proof

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
prof.dr. T. Sminia,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de faculteit der Wijsbegeerte
op maandag 23 mei 2005 om 13.45 uur
in het auditorium van de universiteit,
De Boelelaan 1105

door

Petrus Theodorus Maria Rood

geboren te Bovenkarspel

Promotor: prof.dr. W.R. de Jong

Voor Rosalie

Die Mathematik wie jede andere Wissenschaft kann nie durch die Logik allein begründet werden; vielmehr ist als Vorbedingung für die Anwendung logischer Schlüsse und für die Betätigung logischer Operationen uns schon etwas in der Vorstellung gegeben: gewisse Außerlogische konkrete Objekte, die anschaulich als unmittelbares Erlebnis vor allem Denken da sind.

David Hilbert

Acknowledgements

The project of which the present study forms the product originally started as an historical investigation into various conceptions of (rational) "intuition" in the philosophy of mathematics, and especially into the role of intuition in mathematical proof. When the project went on, I could not resist thinking about more systematic issues, which is where the heart of philosophy is. This is where I am now, a good point to pay my debts.

Thanks to Wim de Jong, for having given me the opportunity to do this project in the first place, and for having always carefully and critically read the myriad of work I wrote.

Part of the project I stayed at Notre Dame University. I thank Michael Detlefsen, Patricia Blanchette of the Department of Philosophy, and the members of the Logic Group at the Department of Mathematics for providing a stimulating day-to-day intellectual environment. My stay at Notre Dame was generously supported by NWO (Netherlands Organization for Scientific Research). GLOBE and the Algemeen Steunfonds, both from the Free University Amsterdam (VUA), also provided financial support, which is kindly acknowledged.

Many ideas occurring in this study where first presented at conferences and seminars at the Free University Amsterdam, Leyden University, the Center for Logic and the Philosophy of Science at Brussels University (VUB), and the Institute for Logic, Language and Computation at the University of Amsterdam. Thanks to the audiences for their comments and questions.

On various different occasions, I had eye-opening, provocative, clarifying, informative, or otherwise pleasant discussions with several different people: Mark van Atten, Igor Douven, Yuri Engelhardt, Eduard Glas, Bob Hale, Richard G. Heck Jr., Roman Murawski, and Dennis Potter.

Thanks to the reading committee: Lieven Decock, Bjørn Jespersen, Michiel van Lambalgen, Göran Sundholm, and René van Woudenberg.

Göran Sundholm read parts of an earlier draft and provided many useful suggestions for improvement. René van Woudenberg pointed out several errors in the pre-final version. I am the only one who is responsible for the ones that have remained.

I spend good times with the "Sources of Knowledge" group at the Department of Philosophy (Free University, Amsterdam), intellectually or otherwise. Those I haven't mentioned yet are: Arianna Betti, Martijn Blaauw, Erik Kreiter, Karel Mom, Ernst-Otto Onnasch, Cornelis van Putten, Sabine Roeser, and Mariëtte Willemsen.

Johan Rebel often assisted with computer problems.

Jean-Paul Muis of Wöhrmann Print Services provided kind and patient advice during the preparation of the manuscript.

Thanks to Sabina Leonelli and Nikki Smaniotto, for the drinks.

Contents

5. The synthetic a priori in mathematics reconsidered

Chapter 1

Introduction

§ 1.1. Proof in action, according to Kant

The present study is concerned with certain issues turning on *how* a mathematician proves (or demonstrates) a theorem when he proves it. In other words, our interest goes to aspects of the procedure or method of mathematical proofs, or proofs in mathematics, if you prefer.[1] As the subtitle of this thesis indicates, we will concentrate on the views of that one great Königsberger philosopher: Immanuel Kant.[2] But before we definitely state our aim, let us first warm up and briefly look at an example.

Consider the following well-known theorem from elementary plane geometry:

THEOREM 1. *The sum of the internal angles of every triangle is equal to two right angles*.[3]

In the following striking passage from the *Critique of pure reason* [94], Kant describes in considerable detail how he thinks a (or any) mathematician ideally proves this theorem (we present some elucidation shortly):

> He [i.e., a mathematician] begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of this triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he

[1] It is not clear whether there is such a distinct kind of thing as *mathematical* proof, as opposed to other kinds of proof. In chapter 3, however, we shall see that Kant believed that mathematical proof is indeed a distinct kind of proof. See also § 5.3.

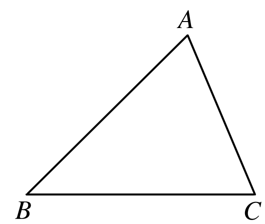
[2] More precisely, we concentrate attention on Kant in his so-called *critical period*, that is, the Kant of the *Critique of pure reason* and later works.

[3] The sum of two right angles is equal to 180°.

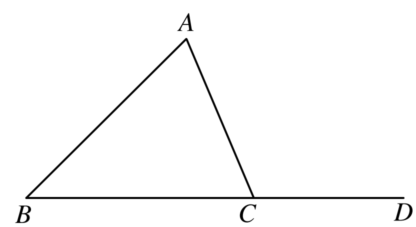
> arrives at a fully illuminating and at the same time general solution of the question (A716-7/B744-5).[4]

In this study, we frequently return to this quotation. Though the proof contained in it is not very interesting mathematically speaking, we nevertheless think it forms a rich source of insights into Kant's views. For convenience, we henceforth refer to it as *The Passage*.[5]

Kant says that the first thing a mathematician does in order to prove theorem 1 is to construct a triangle. In order to fix our thoughts, let us assume that this construction produces a triangle as drawn below:



Subsequently, Kant goes on, the geometer extends one side of this triangle. He does this in view of his knowledge that the angles that can be drawn at one point on a straight line together are equal to two right angles.[6] Let us assume that he extends side *BC* to a point *D* (say), as follows:



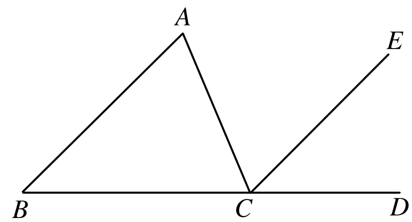
[4] In referring to Kant's *Critique of pure reason*, we adopt the customary habit of using the page numbering of both the A and B edition. Only references to the *Critique of pure reason* won't contain a pointer to the relevant item in the list of references at the end of this study. Thus, an expression of the form A*m*/B*n* refers to page *m* of the A edition and page *n* of the B edition of the *Critique of pure reason*. (In case a page does not occur in the A edition or not in the B edition (which occasionally happens), we shall locate that page by way of an expression of the form A*n* or B*n* respectively.) Quotations from the *Critique of pure reason*, as well as those from any other work of Kant, are all taken from the relevant volume in "The Cambridge edition of the works of Immanuel Kant." (The volume containing the *Critique of pure reason* has both the page numbering of the A and the B edition.)

[5] In chapter 4 (especially § 4.2), we provide a detailed analysis of The Passage, using insights obtained in chapter 3 and the framework presented in § 4.1.

[6] What Kant evidently has in mind is that these angles are to be drawn at a point on a straight line, but all on the same side of that line.

By extending the side of the triangle as shown in the diagram above, the mathematician obtains two adjacent angles, namely, $\angle ACB$ and $\angle ACD$. He concludes that the sum of both these angles is equal to two right angles.

Next, he divides the external angle (i.e., $\angle ACD$) by drawing a line CE (say) parallel to the opposite side of the triangle. Assume that he draws this line thus:



Accordingly, there arises an external adjacent angle, say, $\angle ACE$.[7] The mathematician concludes that this angle is equal to an internal opposite angle, i.e., $\angle BAC$. *Et cetera*.

Here Kant's description of the proof of theorem 1 stops. In order to complete the proof, we may nevertheless presume that the mathematician would continue as follows. He concludes that there arises a second external adjacent angle, $\angle DCE$. The latter is equal to the other internal opposite angle, $\angle ABC$. Since the sum of the two external angles $\angle ACE$ and $\angle DCE$ and the adjacent internal angle $\angle ACB$ is equal to two right angles, he therefore finally concludes that the sum of the internal angles $\angle BAC$, $\angle ABC$ and $\angle ACB$ is equal two right angles. This is what had to be proven.

§ 1.2. Aim and scope

The goal of this study is to throw new light on Kant's views regarding certain aspects of the methodology of mathematical proofs, and to reevaluate Kant's views on mathematical proof accordingly (see § 1.3).

Owing to the complexity of the issues concerned, coupled with limitations of space and time, we mainly restrict ourselves to proofs from elementary Euclidean geometry.[8] Thus, the reader should bear in mind that whenever we speak of mathematics and related issues, we always mean mathematics as restricted accordingly, unless otherwise stated. In particular, we do not enter into Kant's views on the methodology of proving theorems in algebraically oriented parts of

---

[7] In fact, two external angles arise.

[8] Say, the geometry as it is more or less practiced in Euclid's time-honored *Elements* [41].

mathematics (cf., e.g., A717/B745). Our conclusions are accordingly not meant to apply there.

We present two examples of proofs from modern general topology in an appendix (these proofs have a strong geometric flavor). Thus, we indicate that Kant's methodological views may very well apply beyond the mathematics of his time.[9]

§ 1.3. A question

In regard of The Passage, Kant gives us the impression that he thinks of a mathematical proof primarily as a certain *cognitive procedure* carried out by a competent mathematician. Perhaps we may typify the proof Kant describes in The Passage as a kind of *mental animation*. A diagram is being created and is subsequently modified for several times (by adding lines). Furthermore, it seems that the creation and the successive manipulations of the diagram form the primary means for the inferences made. A goal of the proof Kant describes is to prove the truth of theorem 1, and hence to get to know that theorem (cf. A734/B762). We are interested in a certain aspect or feature of this procedure, which we bring to attention by posing a specific question.[10]

Reconsider a mathematician proving a theorem as described in The Passage. Distinguish between what this mathematician reasons *with* from what he reasons *about*. Concentrate on the former and not on the latter. In general, a mathematician proves his theorems with may be called *knowledge* (broadly understood). Now, those concerned with the study of proof typically split this knowledge into distinct "knowledge quanta" or, as we henceforth tend to say, *items of knowledge*.[11] A natural question is the following:

> *what type (or types) of item of knowledge does a mathematician employ when he proves a theorem?*

We approach Kant's views on the method of mathematical proof with this specific question in mind, and the bulk of this thesis (especially chapters 3-5) can

[9] Compare Friedman [49], pp.xi-xiii, who holds that Kant's entire philosophy of science, and his philosophy of mathematics in particular, was intimately related to the state of the art of science in Kant's time, and must nowadays be considered out of date (see also § 1.4). However, Friedman also sees positive value in Kant's thought. Precisely because Kant was so well acquainted with the science of his days, his views would stand as a model for contemporary philosophy (*ibid*., p.xii).

[10] Note that our focus on the cognitive dimensions of Kant's views on mathematical proof finds clear motivation in the fact that Kant was above all a transcendentalist philosopher. Accordingly, Kant sought to account for knowledge by considering the cognitive procedures an idealized agent carries out in consciousness in order to get that knowledge (cf. Posy [127]).

[11] The product of a proof—i.e., the theorem proved by it—is also an item of knowledge. See § 2.2 for further discussion.

be considered what we think Kant's extended answer to it would be. We outline the considerations leading up to this answer shortly.

Nowadays, a quite common answer to the above question is that a mathematician proves a theorem by employing propositional items of knowledge: a mathematician proves a theorem by inferring propositions from other propositions by applying logical rules of inference to them. Accordingly, a proof can be exhaustively represented or formulated in terms of language (sentences). We will refer to this type of reasoning as *propositional reasoning*, thus suggesting a specific view on the structural organization of a proof.

An answer along these lines is intimately related to views on mathematical proof arising from modern logic. What we particularly have in mind is logic as it is conceived in the tradition stemming from Frege and the logical empiricists. One of the more fundamental presuppositions underlying this tradition is that a mathematical proof can be exhaustively formulated in a language. We add that, within this tradition, logic is often taken to be in close association with scientific methodology. See chapter 2 for further discussion.

In the past, however, other answers have been given to the question stated above. For example, a mathematician may prove a theorem by employing concepts, or ideas. Something along these lines can be found in the thought of, for example, Descartes, Leibniz, Locke, and Hume, among others. Interestingly, up to varying degrees, these authors furthermore manifest a critical attitude with respect to the relation between logic (i.e., syllogistic logic) and mathematical methodology. In particular, they see little or no value in logic as an instrument for proving theorems[12] (cf., e.g., Descartes [36], pp.36-7; Leibniz [102], pp.476-8; Locke [107], pp.669-77). We discuss Locke's views in § 3.1. Let us now briefly turn to Kant.

To be somewhat more precise, Kant acknowledged two fundamental types of items of knowledge: concepts and intuitions.[13] The one that plays its distinctive (§ 3.3) role in mathematics—and mathematical proof in particular—is, in Kant's view, the intuition.[14] With regard to his predecessors, the notion of intuition seems to be a novel element of Kant's thought. It forms one of the central elements of Kant's philosophy as a whole, and is of vital importance for his views on the mathematical method of proof.

Kant characterizes an intuition as an item of knowledge that is (1) immediate and (2) singular. It is not readily apparent how these two characteristics are to be understood. Given the importance of Kant's notion of intuition for his philosophy

---

[12] Indeed, their attitude was critical with respect to the relation between logic and scientific methodology in general.
[13] The proposition (judgment) he considered as a derived item of knowledge: a proposition is built from concepts, intuitions, or both.
[14] Kant's original German term is *Anschauung*. In fact, Kant distinguished between several types of intuitions, among which are intuitions a posteriori and intuitions a priori. The ones playing their distinctive role in mathematics are intuitions *a priori* (see § 3.2 for further discussion).

of mathematics, the issue has caused much debate. We review the most central contributions to this discussion, point out the weak spots, and suggest better alternatives in their place. See § 3.2.

Kant, it turns out, holds that an intuition is to a great extent constituted by relations in space (and time[15]). Accordingly, an intuition is an item of knowledge organized in space and time. This suggests that an intuition is an item of knowledge of a quite specific format. We propose to construe an intuition as a diagrammatic item of knowledge (§ 3.4). That said, we can now provide a brief sketch leading up to what we think is Kant's answer to the above question—or so we shall argue. Details will be provided mainly in chapters 3-5.

According to Kant, a mathematician, *qua* mathematician, essentially proves his theorems by way of a distinctly mathematical procedure, or method. This method, we will argue, cannot be accounted for by (general) logic alone. For Kant, the mathematical method of proof is fundamentally constructive, meaning that a mathematician proves a theorem by way of constructing concepts. Construction is a feature of a distinctly mathematical method, which, in Kant's view, may be properly called a special logic of mathematics. See § 5.3.

To construct a concept means to exhibit a priori the intuition corresponding to that concept (§ 3.3). In view of things said earlier, it follows that, according to Kant, a mathematician constructs his concepts (e.g., the concept of a triangle, or a line) diagrammatically, in terms of intuitions. A careful analysis of The Passage (to be carried out in § 4.2) will lead us to conclude that, in Kant's view, the reasoning a mathematician undertakes turns on the spatial relations constitutive for an intuition. Consequently, a mathematician employs intuitions when he proves a theorem. This provides the answer to the question posed above.

As a result, note that Kant's notion of intuition now comes to stand in an interesting new[16] light. Our approach suggests that it can be typified somewhat as follows: an intuition constitutes a specific, i.e., diagrammatic, mode of cognitive organization. In the light of this, mathematical reasoning is in Kant's view a form of diagrammatic reasoning[17]—indeed, *essentially* so. This, we think, implies that Kant does not accept that a proof can be exhaustively represented in terms of a language.

---

[15] We will not consider these temporal relations and mainly consider the spatial relations constitutive for an intuition. In Kant's view, temporal relations play an important part in inferences involving continuity. See Friedman [49], chapter 1, and especially pp.71-80, for a discussion.

[16] We disagree with Hintikka and Beth, who suggest that an intuition comes very close to what logicians would nowadays call an individual constant (or singular term). See § 3.2 for further discussion.

[17] Something along these lines was also proposed by Thompson [156], p.100, but for somewhat different reasons. According to Thompson, Kant's point that mathematical proofs are *demonstrative* (i.e., that mathematical proofs *show*, or make one *see* the truth of a theorem; cf. A735/B763) can be explained by posing that mathematical proof are in Kant's view diagrammatic in nature. However, since Thompson did not pursue his proposal in any detail, it is hard to see what it comes down to.

§ 1.4. Motivation

Why is it interesting to confront Kant with the question posed in the previous section and to undertake an attempt to find out what Kant's answer to it will be?

The question from § 1.3 evidently has historical interest. By trying to find out what Kant's answer to it will be, we isolate an aspect of his views on mathematical proof, and increase our understanding of it. Furthermore, and perhaps more importantly, we will be thus able to correct a misinterpretation of Kant's views due to Hintikka [72], [73] and Beth [16], [17], [18]. Both Hintikka and Beth believe that there is no deep conflict between Kant's conception of proof on the one hand and a modern, logical conception of proof on the other. In fact, Hintikka and Beth seem to believe that Kant's views on mathematical proof can be adequately interpreted (or reconstructed) in terms of systems of natural deduction (see *ibid.*).

A reading of Kant as advocated by Hintikka and Beth strongly suggests that Kant is someone according to which mathematical proof is a form of propositional reasoning. Related to this, we would be committed to believe that, in Kant's view, a mathematical proof can be entirely represented in terms of a language, which we think is highly problematic. Furthermore, if we follow the Hintikka-Beth reading of Kant, an important point is not thematized and accordingly swept under the carpet, namely, the relation between the mathematical method of proof on the one hand and logic on the other. Hintikka and Beth appear to assume without much ado that the method of mathematical proof is essentially the method of natural deduction. For Kant, however, the relation between logic and the method of mathematical proof is a far from trivial issue, and he made a couple of pertinent distinctions on this score. See § 5.3; cf. also § 2.3.

Intimately related to the previous points, there are clear systematic interests too. In contradistinction with a view as expounded by Beth and Hintikka, it is sometimes also held that developments in modern logic have made Kant's views on proof obsolete.[18] For example, a few paragraphs after quoting The Passage, Michael Friedman repudiates Kant's views on mathematical proof in fairly strong language:

> Kant's conception of geometrical proof is of course anathema to us. Spatial figures [i.e., diagrams], however produced, are not essential constituents of proofs, but, at best, aids […] to the intuitive comprehension of proofs (Friedman [49], p.58).

---

[18] Another reason for downplaying Kant's views on mathematics (and geometry in particular) turns on Kant's (supposed) views on the geometry of physical space in combination with certain developments in physics. In this respect, we can especially mention the rise of theory of general relativity at the beginning of the 20th century. Cf. Friedman [49], p.340-1. See also Reichenbach [133], especially p.6.

On the positive side, Friedman holds that a

> […] proof […] is a purely "formal" or "conceptual" object: ideally a string of expressions in a given formal language (*ibid*.).

In his rather negative assessment[19] of Kant's views on mathematical proof, Friedman [49], p.56, follows Russell [137], p.457, [138], p.145. See also Ayer [1], pp.110-1, for a statement of a view similar to that of Friedman.

As with Hintikka and Beth, Friedman's conception of proof is evidently one taking its orientation from modern logic. However, as suggested, Friedman's evaluation of Kant goes in an almost complete opposite direction. While Beth and Hintikka offer room for a vindication of Kant's views on this score (cf. Hintikka [76], pp.174-98), such is not the case with Friedman. According to Friedman's own views, a mathematical proof can be represented exhaustively in terms of language. In fact, according to Friedman, a mathematical proof ideally *is* a sequence of sentences. The impression that is thence forced upon us is that, in Friedman's view, mathematical reasoning would be a kind of propositional reasoning, namely, reasoning with the propositions expressed by those sentences. Friedman's own views on mathematical proof, furthermore, seem intimately related to a methodological conception of (modern) logic (cf. Friedman [49], p.58).

Besides the fact that Kant would not accept that mathematical proof is a form of propositional reasoning (see above), there is no reason to believe that *we* should. Again, we think that we touch here upon a presupposition of a logic-oriented conception of proof, a presupposition that may be legitimately put into question.[20]

We think that on the whole Friedman's negative assessment of Kant's conception of proof is somewhat exaggerated. More positively, we believe that there is, *grosso modo*, nothing intrinsically wrong with the proof Kant describes in The Passage. Quite the contrary, we are strongly inclined to think that Kant

[19] But see also footnote 9.

[20] Though we do not intend to make Kant a spokesman for modern discussions, his views in this respect evidently raise issues that are of contemporary relevance. For example, questions concerning the format of knowledge have always been acknowledged to be of fundamental importance in disciplines such as cognitive science and Artificial Intelligence (AI). For example, one only needs to consider the connectionism debate in cognitive science. Furthermore, a knowledge representation system, as it is typically understood within AI, has at least two components. First, a knowledge base consisting of a set of "data structures" in terms of which knowledge is represented. Second, an associated "inference engine" that allows the system to execute inferences over the data structures in the knowledge base. Those concerned with the theory and design of knowledge representations systems have considered various ways of storing the knowledge in a knowledge base, varying from propositional formats to, for example, semantic nets or frames. See also van Benthem, [14], p.10, who stresses that formatting issues are important for logic as well; cf. also van Benthem [13], p.292.

has in fact given us a very appealing description of a proof of theorem 1 (cf. Beth [18], p.45). This, we think, gives Kant's conception of proof a considerable degree of credibility.

It thus turns out that the logical conception of proof plays a quite pivotal role in modern interpretations and evaluations of Kant's views on mathematical proof. We believe that the Beth-Hintikka reading is incorrect and that Friedman's negative assessment is not entirely justified at the same time. The question "what type of items of knowledge does a mathematician view reason with when he proves a theorem?" forms one means to pinpoint our dissatisfaction on both sides.

We look at Kant as someone who has deep and still valuable insights into the cognitive and methodological dimensions of mathematical proof, though they often need not accord well with modern logical conceptions of proof. Kant's point that a mathematician essentially reasons with intuitions is intimately related to this. Accordingly, we think it is incorrect to reject Kant's views on mathematical proof because they do not seem to accord well with a modern logical conception of proof, as, for example, Friedman does. In contrast, we believe that Kant's views on proof have to be looked at from a wider perspective. For Kant, besides logic, mathematical proof involves elements of a cognitive nature as well.[21] Thus, what we nowadays call logic and psychology are in Kant's view more tied together than they often appear today.[22] Furthermore, in Kant's views, the procedure a mathematician executes when he proves a theorem is not just a matter of logic. It crucially involves considerations turning on a distinctly mathematical method as well, thus making the relation between logic and philosophy of science considerably more complex.

§ 1.5. Outline

In outline, our study takes the following form.

Chapter 2 discusses the view that a mathematical proof is ideally a logical proof, a view that is pivotal in modern interpretations and evaluations of Kant (§ 1.4). The subsequent chapters are devoted to our main task: the systematic exploration of Kant's views on the methodology of mathematical proofs.

---

[21] It is of some interest to note that there is nowadays a growing attention for reasoning with diagrams (or diagrammatic reasoning) from cognitive scientists and scholars from the AI community. See, for example, the volume edited by Glasgow, Narayanan and Chandrasekaran [58]; see also Kulpa [98], including the references found there. Recently, logicians, too, have shown interest. See, for example, Shin [145], and Hammer [62]; cf. also the previous footnote; cf. also § 2.1.

[22] See also § 2.4. Let us add that Kant was not a psychologist. The proof described in The Passage forms as much a rational reconstruction as a logical proof is supposed to do (cf. § 2.4): it is a proof carried out by some idealized agent.

Chapter 3 discusses the two essential components of Kant's views on the mathematical method of proof:[23] his notion of construction and the related notion of intuition.

In chapter 4, we shall begin by presenting a methodological framework for mathematical proofs (§ 4.1). The rest of this chapter will be devoted to a detailed analysis of the methodology that we think lies at the background of The Passage. In our analysis, we will use insights obtained in chapter 3 and § 4.1.

In chapter 5, we readdress a central issue for the philosophy of mathematics, namely, Kant's views on the nature of the synthetic a priori.

We close off by stating our conclusions.

[23] For Kant, construction and intuition not only play their role in mathematical proof but within mathematical science generally. See § 3.3.

Chapter 2

Logical ways: proof by natural deduction

In the present chapter, our attention goes to what we call the *logical conception of proof*. Generally put, according to the logical conception of proof, a mathematical proof is (ideally) a logical proof. In various different though related forms, the logical conception of proof was prominent in Frege and Russell, among others, who can be reckoned among the founding fathers of modern logic.[24] Via logical empiricism, it has subsequently found a firmly established place in today's philosophical thinking about mathematical proof (§ 2.1). The main purpose of this chapter is to characterize the logical conception of proof and to bring out some fundamental assumptions underlying it.

Why it is interesting to delve into issues underlying the logical conception of proof within the context of a study on Kant? As we have seen (§ 1.4), the logical conception of proof has heavily influenced the way Kant has been interpreted and evaluated. This forms ample reason to consider it somewhat more closely. Accordingly, we can clean up the way for a more adequate reading of Kant as well as to repave it in order to obtain a more balanced evaluation of his views on mathematical proof; hence, the present chapter.

We begin with a general characterization of logical proofs (§ 2.1). Second, we bring out an important assumption underlying the logical conception of proof (§ 2.2). Third, we distinguish two respective types of logical inference used in logical proofs, and discuss an interpretation of The Passage due to Beth which has formed the basis for Hintikka's reading of Kant (§ 2.3). Finally, we critically discuss important distinction from the philosophy of science that has motivated the logical conception of proof, namely the distinction between "context of discovery" and "context of justification" (§ 2.4).

---

[24] In case of Frege, this conception of proof functioned in the so-called logicist program (with respect to arithmetic; see Frege [45], [46]). The logicist program puts certain strict requirements on the axioms, definitions, and rules of inference figuring in a logical proof. First, the axioms should be logical truths. Second, a definition should define its definiendum in terms of logical definientia. Third, the rules of inference should be logical rules of inference. From the point of view of the logical conception of proof, we may say that logicism is especially concerned with the logical status of axioms and definitions. As regards the logical conception of proof, we shall only be concerned with logical rules of inference (and not with the status of axioms and definitions), hence, the present chapter sets specific logicist concerns aside.

§ 2.1. Logical proofs

Logical proofs characterized. It may be said that one of the aims of logic is to define and study consequence relations between sets of sentences and, in many cases, individual sentences (cf. Gabbay [52]). Let $\vdash$ be such a relation, and, where Γ is a set of sentences and ϕ an individual sentence, let $\Gamma \vdash \phi$ (read as: "ϕ is a consequence of Γ"). In general, a definition of $\vdash$ can be given from the point of view of proof theory or from the point of view of model theory. From the point of view of proof theory, we say that ϕ is a consequence of Γ if there is a logical proof of ϕ from Γ. From the standpoint of model theory, we say that ϕ is a consequence of Γ if ϕ is satisfied by every model that satisfies every sentence in Γ. We focus on the proof theoretic standpoint, since this is the one most relevant for our purposes. In the light of this, we will henceforth say that a sentence is *provable* from a set of sentences instead of being a consequence of it.

Given our proof theoretic standpoint, a logical proof is always presented or formulated relative to a logical system for short (or a "logic").[25] A logical system is given by "specifying":

- a language;
- a collection of rules of inference.

In general, relative to a logical system, a logical proof is presented or formulated as a certain finite configuration (e.g., a sequence, or a tree) of sentences. Below we will offer a more specific characterization of logical proofs. Let us first settle a few preliminary conceptual points.

A *language* is a set of sentences. A sentence, in turn, is a meaningful unit of expression. We assume that a sentence is always a declarative sentence. Examples of sentences are:

- *two points determine exactly one line*;
- *every finite straight line can be bisected*;
- *the sum of the internal angles of every triangle is equal to two right angles*.

We mention specific sentences in the usual sloppy way, by putting them in *italics*.[26] Occasionally, we mention sentences by putting them between 'single quotes.' Similar conventions hold for every other expression (e.g., an individual constant (singular term), a predicate, etc.). When we refer to a sentence without having any specific sentence in mind, we typically use Greek letters such as ϕ,

[25] Barwise and Feferman [9] consider logical systems more from the standpoint of model theory.
[26] Italics will be used for other purposes as well, e.g., in order to emphasize.

ψ, etc. Occasionally, we also use letters from the Latin alphabet, e.g., p, q, etc. for the same purpose.

The above examples are sentences of a natural language, i.e., English. However, logicians do typically not consider natural languages but certain artificially designed languages instead. With respect to natural language, these languages are typically used to exhibit certain logically relevant features of the sentences that form the object of logical scrutiny, for example, logical form. For instance, in a standard first-order language, the sentence *two points determine exactly one line* can be paraphrased as:

$$\forall x \forall y \exists z \forall u((point(x) \wedge point(y) \wedge line(u, y, x)) \leftrightarrow u = z).$$[27]

See § 2.2 for a discussion of two of such artificial languages.

A *rule of inference* can be seen as a license to carry out an inferential step, i.e., a license to infer one sentence using several others. Consider, for example, the well-known (logical) rule of inference known as *modus ponens* (typically figuring in Hilbert-style systems or natural deduction systems—see below). According to this rule, one is allowed to infer, for instance, the sentence *the sum of ABC's internal angles is equal to two right angles* from the two sentences *if ABC is a triangle, then the sum of its internal angles is equal to two right angles* and *ABC is a triangle* (see also § 2.3).

Given a rule of inference, we generally say that an inference is carried out *in accordance with* the rule. Alternatively, we sometimes say that a sentence is inferred by *applying* a rule. Note that in both cases, our attention is accordingly drawn to an inferential *procedure* or *process*. We will return to the procedural dimensions of logical inference in § 2.3.

Many different types of logical systems have been considered by logicians, each of them determinative for a certain "logical proof style." Let us mention some of these systems, without pretending to have given a complete list:

- Hilbert-style systems;
- natural deduction systems;
- sequent-style systems;
- resolution systems;
- tableaux systems.

For example, a logical proof in a Hilbert-style system is referred to as a *Hilbert-style proof*; a logical proof in a natural deduction system is referred to as a *natural deduction proof*, etc. An interesting question is whether any of these proof styles can be taken to correspond to the way a mathematician would

---

[27] *line*(t, v, w) means "t is a line determined by v and w"; the predicate *point* speaks for itself. We take a sentence $\phi \leftrightarrow \psi$ to abbreviate the sentence $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

(ideally) prove a theorem. The answer to this question is often taken to be affirmative. For example, Gentzen, who was among the pioneers of the study of natural deduction, believed a system of natural deduction "reflect[s] as accurately as possible the actual logical reasoning involved in mathematical proofs" (Gentzen [56], p.291; cf. Barwise and Hammer [10], p.77).

Gentzen's point is far from obvious, however. In particular, no clear criteria for "accurateness" are provided. Furthermore, it may even be said that there is plenty of *prima facie* evidence counting against it. For example, proofs as they are written up in mathematics books and journals do not appear to be natural deduction proofs. One may point out that these proofs can always be turned into natural deduction proofs by filling in the extra steps that are sometimes left out. Now it is indeed true that mathematicians often deliberately leave certain steps as an almost proverbial "exercise for the reader." However, it is far from clear whether filling in those steps would result in a natural deduction proof in the end. On the contrary, it may very well be that a proof merely becomes more long-winded but no less close to a natural deduction proof, or, for that matter, any other type of logical proof (e.g., a Hilbert-style proof).

Further, to say that a natural deduction proof accurately (or at least as accurately as possible) reflects the procedure carried out in a mathematical proof—which is what Gentzen seems to have in mind—turns out to have a significant consequence. For according to such a view, natural deduction proofs are not merely taken as means to define a purely extensional relation of provability. In contrast, they also provide "intensional" information turning on the procedures a mathematician carries out when he proves a theorem. Put differently, natural deduction proofs are also supposed to reflect a certain method, i.e., a method of mathematical proof. As a consequence, systems of natural deduction turn out to have clear methodological dimensions too, making them suitable as a topic in a chapter of the philosophy of science.

The logical conception takes natural deduction proofs—or, more generally, systems of natural deduction—in this methodological sense. According to the logical conception of proof, a mathematical proof can be formulated or represented as a certain configuration of sentences, which accordingly reflects a method of proof. We may refer to this method as the method of natural deduction.

Let us add that not every type of logical system together with its accompanying proof style is taken in this sense (cf. Barwise and Hammer [10], p.77). For example, in contradistinction with natural deduction systems, Hilbert style systems provide a theoretically elegant characterization of provability but it appears that such systems fail to reflect the structure of mathematical proofs. A similar point holds for sequent-style proofs, which are often invoked for the mathematical study of the provability relation itself. Resolution or tableaux style proofs, finally, are considered because they have properties making them particularly suitable for implementation on a computer. Again, however, such systems fail to reflect adequately the structure of the proofs as given by mathematicians.

Henceforth, we shall restrict ourselves to systems of natural deduction: a logical proof is always natural deduction style, and hence presented relative to a system of natural deduction. The reason for this choice should be obvious by now. We shall also assume that logical proofs in such systems are presented as sequences of sentences.[28] We refer to the final sentence of a logical proof (in a system of natural deduction) as a *theorem*. Some of the sentences constituting a logical proof are called *premises* (or axioms—see below). For theoretical elegance, we do allow cases where a theorem is a premise.

What is distinctive for systems of natural deduction, as opposed to other types of systems, is the possible use of assumptions (cf. Prawitz [128], p.23, incl. n.1): some of the sentences cited in a logical proof, except for the premises and the theorem, are allowed to be assumptions. For the moment, assumptions can be seen as auxiliary sentences introduced in the course of a proof (if only temporarily) in order to infer other sentences. When a sentence has been inferred using assumptions, these assumptions need to be properly accounted for by "discharging them." See § 2.3 for more details.

We assume that the language of a system of natural deduction is a first-order language.[29] It should be added, however, we are not so much interested in first-order languages *per se*. The reason for our choice is mainly that it gives us clear footholds on the specific types of inferences that are allowed relative to such systems.

A rule of inference of a system of natural deduction is sometimes called a *rule of natural deduction*. In the present section, it is not important to know what the rules of natural deduction precisely are. They are more extensively discussed in § 2.3.

From the point of view of logic, there is a strict separation between the "inferential regime" of a mathematical proof on the one hand and the mathematics on the other. The inferential regime is accounted for by a logical system: it determines what sentences can be legitimately inferred using others and how. However, an inferential regime alone does still not give us mathematical proofs. From the standpoint of a logical system, mathematics comes in at the axioms. Let us present the following definition.

Let S be a logical system (i.e., a system of natural deduction). A *theory* in S (or simply a *theory*) is a subset of the language of S. We think of the members a theory as representing the axioms of a branch of mathematics. For example, a theory in some logical system may represent a set of axioms for Euclidean

---

[28] Some logicians tend to define a logical proof not as a *sequence* of sentences but as a *tree* whose nodes are labeled with sentences instead (cf. Prawitz [128], van Dalen [32]). However, this difference is not a substantial one but merely concerns two different forms of notation.

[29] We restrict ourselves to the following logical constants: $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material conditional), $\neg$ (negation), $\forall$ (universal quantifier), and $\exists$ (existential quantifier).

geometry.[30] The axioms of a logical system do not admit of a proof (in that system), except a trivial one-line proof. In the present study, we assume that every premise cited in a logical proof is an axiom. See below for a further discussion on the nature of axioms. We present the following definition:

DEFINITION 1. Let S be a system of natural deduction and let $\mathscr{L}$ be the language of S. Let Γ be a theory in S. A Γ-*logical proof* in S is a finite sequence of sentences of $\mathscr{L}$ such that every sentence in the sequence is an axiom in Γ, an assumption, or is inferred by means of the application of a rule of natural deduction, using earlier sentences in the sequence.

A definition along the lines of definition 1 has found a firmly established place in logic textbooks. See, for example, Mates [109], p.113, 166, 180; Barwise and Etchemendy [8], pp.48-9; Tidman and Kahane [157], p.42; Bonevac [19], p.107. Many more references could be added to this list.

As it stands, definition 1 is not fully precise. In particular, it is not clear how one should take account of the assumptions, which involve some intricacies. The point can only be settled after the rules of natural deduction are specified (§ 2.3).

Instead of Γ-*logical proof in* S, we henceforth often simply use *logical proof*, unless confusion is possible. However, whenever we speak of logical proofs, the reader should bear in mind that we always presuppose *some* logical system and *some* theory in that system.

Let Π be a logical proof. Without loss of generality, we henceforth assume that all the axioms are cited in an initial fragment of Π. Accordingly, where $\alpha_1,\ldots, \alpha_k$ are the axioms cited in Π, and $\phi_{k+1},\ldots, \phi_n$ are the remaining sentences, Π can be written as

$$\alpha_1,\ldots, \alpha_k, \phi_{k+1},\ldots, \phi_n.$$

We say that Π is a *logical proof of* (*the theorem*) ϕ_n *from* the axioms $\alpha_1,\ldots, \alpha_n$. Alternatively, when Γ is the underlying theory (so that $\alpha_1,\ldots, \alpha_n \in \Gamma$) we say that Π is a *logical proof of* ϕ_n *from* Γ.[31]

Insofar as the sentences constituting a logical proof are concerned, we can distinguish between sentences that are inferred and sentences that are not inferred. The sentences that are not inferred are precisely the axioms and the assumptions. See definition 1. It will turn out that the inferred sentences come in two different types: sentences that depend on an assumption and sentences that do not depend on an assumption (see § 2.3). The final sentence of a logical proof

[30] We ignore definitions, since, in the present context, these are eliminable.

[31] If Π is a logical proof of ϕ from Γ then evidently there are $\alpha_1,\ldots, \alpha_n \in \Gamma$ such that Π is a logical proof from $\alpha_1,\ldots, \alpha_n$.

is not allowed to depend on an assumption. In effect, then, the sentences possibly cited in a logical proof are of four different types: axioms, assumptions, sentences dependent on an assumption, and sentences not dependent on an assumption.

Logical proofs have the following property. Let Π be the following logical proof:

$$\phi_1, \phi_2, \ldots, \phi_k, \ldots, \phi_n.$$

Suppose ϕ_k is a sentence not dependent on any assumption ($k \leq n$). Then the following is also a logical proof, namely, a logical proof of ϕ_k:

$$\phi_1, \phi_2, \ldots, \phi_k.$$

This property allows us to refer to any sentence of Π that does not depend on any assumption also as a theorem. Notice, however, that Π itself is not a logical proof of such a sentence but an initial fragment of Π instead.

We can now restate an earlier point: as to the sentences cited in a logical proof, we can distinguish between four different types, namely (i) axioms, (ii) assumptions, (iii) sentences dependent on an assumption, and (iv) theorems. We need this in the next section.

Logical proofs, truth, and (propositional) items of knowledge. A logical proof, as we have defined it (cf. definition 1 above), is merely a certain sequence of sentences. In particular, notions such as truth and knowledge do not figure in definition 1. As such, a logical proof is not properly speaking a proof: a logical proof on itself does not really *prove* a theorem. The most a logical proof does is to show that a conclusion *is provable from* several axioms.

Now what is a proof? In general, a proof is something that *establishes* the truth of a theorem (cf. also A734-5/B762-3). Once the truth of a theorem has been established, then that theorem is known to be true. Note that this is a *functional* description of proof: a proof is characterized in terms of a certain function it is supposed to fulfill, namely, to establish the truth of a theorem.

Once we have a functional characterization of proof such as the one presented above, we can turn to matters pertaining to the implementation of this functional description. We can turn to questions such as: what procedure or method does one execute in order to establish the truth of a theorem? In order to deal with such questions, many parameters that can be set. The values of these parameters will be strongly dependent on one another. For example, one such parameter turns on the type of item of knowledge employed in a proof, which is what interests us here. However, a decision on this point will certainly influence the type of inferential procedure used in order to process these items of knowledge.

However, one may desire more of a proof than that it merely establishes the truth of a theorem (cf. Rav [132]). For example, one may require of a proof that it yields a certain insight, perhaps into the reason *why* a theorem is true. Further, one may demand that a proof be based on reusable techniques. Other requirements on a proof procedure may turn on the available resources usable in order to prove a theorem (e.g., time and memory space). It is far from clear how these requirements affect the parameters concerning the implementation of a proof. We lay this matter to rest.

As will be clear by now, in order for logical proofs to be proofs in the proper sense of the word, we have to dress them up in terms of two other notions: truth and knowledge. To this we turn shortly. We first settle a few preliminary points.

We can make a distinction between sentences on the one hand, and the propositions they express on the other. For example, we say that the sentence *two points determine exactly one line* expresses the proposition *that* two points determine exactly one line. The proposition expressed by a sentence is an aspect that can be shared by other sentences. For example, the aforementioned English sentence expresses the same proposition as the German *zwei Punkte stellen genau eine Linie*. The proposition expressed by both sentences is the proposition that two points determine exactly one line.

Let ϕ be a sentence and let a be some "agent." In the present study, we are mainly interested in cases where we say things like "a knows *that* ϕ." Knowledge in this sense turns primarily on propositions, and not on the sentences used to express them. In particular, when we say "a knows that ϕ," we mean to say that a knows that proposition expressed by the sentence ϕ is true (in some sense of 'true').[32] We may say that the proposition that ϕ forms the *object* of a's knowledge. If, as a matter of fact, some agent a knows that ϕ, then the proposition that ϕ is an *item of knowledge*. When we want to stress the propositional format of this item of knowledge, we sometimes refer to it as a *propositional item of knowledge*. (It should not be assumed, however, that all items of knowledge are propositional. See § 3.1 and § 3.2; cf. also § 2.2.).

Having straightened out these conceptual issues, we shall henceforth not always strictly keep track of the distinction between sentence and proposition. It is because of this reason that we shall use the terms *proposition* and *sentence* interchangeably. Our main motivation for blurring the distinction between sentence and proposition is that this prevents us from using all kinds of cumbersome formulations. For example, we shall say such things as: "the axioms are true (or known)." Since axioms, as we have introduced them, are sentences of (the language of) a logical system, we should strictly have said: "the propositions expressed by the axioms are true (or known to be true)." Furthermore, and especially from a more procedural point of view, notions such as proof and

[32] In the light of this, we also assume that it is primarily propositions that admit of truth. Derivatively, we say that a sentence is true if the proposition expressed by it is true.

inference appear to apply primarily to propositions. Thus, an agent proves that ϕ is true (ϕ a sentence), and infers the proposition that ϕ. However, sometimes we will simply say that an agent proves ϕ, or that he infers ϕ. The reader should bear in mind that, strictly speaking, these notions turn on propositions. Nevertheless, our formulations, when taken literally, sometime suggest otherwise.

Let us now dress up logical proofs in order to turn them into proofs in the proper sense. We begin by concentrating on matters of truth; next, we turn to knowledge.[33]

1. *Truth.* We need the following definition. Whenever $\Delta \cup \{\beta\}$ is a set of sentences, we define an *argument* as a pair (Δ, β). An argument is written as Δ / β. We say that an argument Δ / β is *logically valid* (or *valid*, for short) if, necessarily, β is true provided all the sentences in Δ are true.

Note that the definition of validity is not very precise according to logical standards. For the concept of validity depends on the semantics of the language from which the sentences constituting an argument are given. However, for the present purposes, the above definition is good enough.

Let S be a logical system and let Γ be a theory in S. We assume that the axioms in Γ are true (see below). We also assume that the system S is *sound* relative to Γ. By this, we mean the following: whenever a sentence ϕ is provable from Γ, then the argument Γ / ϕ is valid.

The validity of Γ / ϕ in a way secures that a logical proof of ϕ from Γ "carries over" the truth from the axioms to the theorem ϕ. Since the axioms are also assumed to be true, the soundness of the underlying system secures that ϕ is true whenever ϕ is provable. In this respect, soundness can be seen as a kind of closure property. In terms of a slogan, we may say that truth is closed under provability from true axioms.

The following remarks are not strictly necessary for the purposes of this study. Nevertheless, they do add to the completeness of our exposition.

As to the truth of axioms, we can distinguish between the following two views:

(i) an axiom is true with reference to a fixed antecedently given subject-matter;
(ii) an axiom determines a class of models such that the axiom is true in any model in the class. More generally, a *set* of axioms determines a class of

[33] For some (e.g., intuitionists) truth and knowledge are much more intimately related than the following considerations suggest.

> models such that, given any model from the class, all the axioms are true in that model.[34]

In the former case, the axioms are supposed to "fit" the subject-matter; in the latter case, the various "subject-matters" (in the form of models) are supposed to "fit" or satisfy the axioms.

A conception of axioms in the former sense seems to be the traditional one. For example, it seems that the subject matter of geometry is traditionally taken to be physical space and in particular the figures contained in it. A conception of axioms in the latter sense seems to be of relatively recent origin. It can be attributed to Hilbert.

In case of axioms in the sense of Hilbert, we need not to decide whether, for example, certain geometrical axioms are true with reference to an antecedently given subject matter (e.g., physical space). In contrast, we can consider different sets of axioms and accordingly study the models that respectively satisfy them.[35] Each set of axioms determines a "geometry." Thus, the possibility arises of studying a variety of different "geometries" instead of "the one true geometry."

The point can be illustrated by means of the well-known case of the axiom of parallels: *through any point of the Euclidean plane not on a line ℓ there is exactly one line that does not intersect ℓ* (i.e., which is parallel to ℓ). Instead of wondering whether this axiom is true with reference to, say, physical space, one takes this axiom for granted, and considers the models satisfying this axiom (together with the remaining axioms for Euclidean geometry).[36] Alternatively, one may also consider the models satisfying a substitute of the axiom of parallels, e.g., *two lines always intersect*, or *through any point not on a line ℓ there is more than one line not intersecting ℓ*. Both these axioms are inconsistent with the axiom of parallels, but consistent with the remaining axioms of Euclidean plane geometry. Accordingly, these two axioms give rise to two respective types of non-Euclidean geometry, namely, elliptic geometry and hyperbolic geometry. We conclude that from a mathematical point of view, axioms in the sense of Hilbert are clearly preferable, since they serve a more fruitful way of doing mathematics.[37]

It is of some interest to add that Kant defines truth in terms of "the agreement of cognition with its object" (A58/B82). Thus, Kant appears to think

---

[34] Within first-order logic, the class of models of a sentence is known as an EC class; the class of models of a set of sentences is known as an EC_Δ class.
[35] In order for a set of axioms to have models, it should be consistent.
[36] It is often and quite simply held that the axiom of parallels is not true with reference to physical space, since the geometrical structure of physical space is non-Euclidean. The point obviously requires some qualification, however. For the non-Euclidean structure of physical space only manifests itself at a very large scale (of light-years).
[37] This does not mean that axioms in the sense of Hilbert are unproblematic philosophically speaking. For example, there are serious questions pertaining to the ontological status of models.

of truth in terms of the medieval idea of *adequatio*. Kant does not, unlike Brouwer and his followers, identify the truth of a proposition with its being proved. Perhaps we may say that for Kant truth is something more fundamental then proof. Proof, in this respect, is a means to an end: again, a proof establishes the truth of a theorem.

2. *Knowledge*. Thus far, axioms are in effect nothing but sentences of the language of a logical system. This is not how the term axiom has been typically understood. In particular, axioms are often seen as having a specific epistemic status.

Let ϕ be an axiom of a logical system. We shall assume that ϕ is known, that is, it is known that ϕ is true. Traditionally, it is often supposed that the knowledge that ϕ (ϕ an axiom) is true needs no other propositions for its justification besides ϕ. For example, ϕ may be supposed to be self-evident. A proposition is self-evident roughly if one assents to its truth as soon as one understands the concepts involved in it. The notion of self-evidence forms a classical foundationalist theme.

It would seem that axioms in the sense of Hilbert (see above) could not be self-evident in this sense. One reason is that self-evidence appears to be an absolute notion. However, if an axiom in the sense of Hilbert can be said to be known, it is not *simply* known but only with respect to a given model, or class of models. For example, consider the geometrical axiom *two points determine exactly one line*. Thought of as an axiom in the sense of Hilbert, we should for example say that this axiom is known to be true in Euclidean space. On the other hand, the very same axiom is not true in elliptic or hyperbolic space. By way of conclusion, let us state that the justification of axioms in the sense of Hilbert presumably has to be analyzed along other than classical foundationalist lines (see, for example, Bernays [15]).

We furthermore assume that an agent *a* knows that ϕ is true whenever *a* has logically proved ϕ from a collection of known axioms. As with soundness, the latter, too, can be seen as a kind of closure property, but this time with respect to knowledge instead of truth. In a slogan: knowledge is closed under proof (not: provability) from known axioms.

Note the difference between "having logically proved ϕ" and the "provability of ϕ." The latter merely refers to the existence of a logical proof of ϕ. The former, in contrast, *also* refers to an agent and a certain procedure carried out by that agent. In § 2.2, we put this procedure under scrutiny.

Recall that besides axioms and theorems, also other types of sentences are possibly cited in a proof, namely, assumptions and sentences dependent on assumptions. Assumptions are typically not items of knowledge.[38] An agent

[38] Though there is no reason of principle not to use a propositional item of knowledge as an assumption.

typically does not know that an assumption is true. Instead, he merely *assumes* it true. Indeed, an assumption need not even *be* true. A similar point holds for sentences dependent on assumptions. These are likewise typically not known to be true. As with assumptions, a sentence dependent on an assumption need not even *be* true. Perhaps we may say that an agent *holds* such a sentence to be true on the condition that an assumption is true. Instead of items of knowledge, we may say that assumptions and sentences dependent on assumptions express *mere* propositions. Before we end up this section, let us readdress another issue.

In the first paragraph of this section, we said that a proof satisfies a certain functional description, namely, a proof establishes the truth of a theorem. In order to turn logical proofs (cf. definition 1 above) into proofs in this sense, some dress-work had to be done (in terms of the notions of truth and knowledge). In sum, we assumed (cf. above):

(1) truth is closed under provability;
(2) axioms are true and known;
(3) knowledge is closed under proof.

Strictly speaking, we now need to verify whether logical proofs dressed up accordingly are indeed genuine proofs. In other words, we need to verify whether (1)-(3) imply that theorems are known to be true. For the sake of argument, we assume that this is indeed the case, i.e., we assume that logical proofs are indeed genuine proofs. Accordingly, logical proofs can be seen as a specific way of *logically implementing* the aforementioned functional description.

Logical proofs and philosophy. We think that a conception of proof taking its orientation from definition 1 is still the predominant one, especially within more philosophical circles. Many appear to believe that a mathematical proof (ideally) is a logical proof. See, for example, Friedman, quoted in § 1.3. Others that can be mentioned are Steiner [151], Chihara [30], and Bonevac [19], especially p.2. Many more names could be added to this list. Among proof theorists, a conception of proof along these lines also has at least some currency. See, for example, Buss [24].

These considerations suggest that the logical conception of proof is a relatively widely accepted within philosophical circles (but see below). The following case forms an illustrative example to consider somewhat more closely. Although the specific points that the example aims to highlight are not particularly relevant for the purposes of the present study, the case clearly shows how specifically logical notions color philosophical reflection on mathematical proofs.

In a recent paper, Don Fallis [43] has pointed out that mathematicians hardly ever publish their proofs in full detail. Instead, Fallis goes on, mathematicians often intentionally leave out several more detailed considerations in order to make their written proofs not unnecessarily lengthy and thus to communicate them more efficiently. True as this may seem, he goes on to articulate this idea by saying that a mathematician typically does not lay down "the entire sequence of propositions in excruciating detail" (*ibid.* p.55). Fallis apparently thinks of a proof essentially as a sequence of propositions. The difference between a published proof and an ideal proof would be that several "chunks" of propositions are left out of the former which would be maintained in the latter. At any rate, Fallis' conception of proof no doubt has its roots in modern logic, and a definition such as definition 1 in particular.

Not everyone within the philosophical community believes that a mathematical proof is logical proof. A notable exception is formed by those who endorse an intuitionist philosophy of mathematics in the line of Brouwer. It will be clear from definition 1 and 2 above that language has a central place in the logical conception of proof. However, Brouwer held that mathematical proof is essentially a "languageless" mental construction. A mathematical proof has essentially nothing to do with language, and neither with logic—or so Brouwer thought. Brouwer considered logic to be a kind of language, which had fundamentally no place in the process of mathematical proof:

> Logic is not a reliable instrument to discover truths and cannot deduce truths which would not be accessible in another way as well. […] Mathematics rigorously treated from this point of view, and deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics (Brouwer [22], p.1243).

The best language can do is to describe certain regularities in mathematical construction processes and to form a medium for reasoning about these construction processes (Brouwer [23], p.99). However, language does not form the medium in which this construction process itself is carried out. Thus, language has only a secondary role. Again, according to Brouwer, the mathematical construction process itself is essentially languageless.

Logical proofs and diagrams: an overview. Some logicians have stretched up the notion of a *logical* proof in an attempt to accommodate for reasoning with diagrams. The strategy is to take the same template for the underlying logical system, namely, a language and a collection of rules of inference. This time, however, *language* is understood in a somewhat wider sense. Thus, the language of a logical system is not only allowed to include sentences but diagrams as well. Let us provide a brief overview of these developments and raise a few issues on behalf of them. We won't go into the details since this would quickly make our discussion unnecessarily lengthy; we simply refer the reader to the literature instead.

Consider any logical system. If the language of the system only consists of sentences, then the system may be called *sentential*. If the language only consists of diagrams, then the system may be called *diagrammatic*. If the system has diagrams as well as sentences among its members, then, borrowing from Barwise and Etchemendy [4], [5], [6], the system may be called, *heterogeneous* (see also Barwise [2]).

A logical proof in a diagrammatic system is a sequence of diagrams such that any diagram in the sequence is a premise or is inferred by means of the application of a rule of inference (see Hammer [62], p.45, for an example). A logical proof in a heterogeneous system is a sequence of sentences and/or diagrams such that any sentence/diagram in the sequence is a premise or is inferred by means of the application of a rule of inference. Let us mention a few important diagrammatic systems that have been developed and investigated by logicians.

Shin [145] presents two systems for Venn diagrams (which she calls Venn-I and Venn-II). Hammer [62] in part builds on Shin's work and presents systems for Euler diagrams, and Pierce's alpha graphs, among other things. Venn diagrams are originally due to Venn [160]. As noted by Shin [145], p.6, Venn intended his diagrams "to be in complete correspondence and harmony with Boolean algebra" (cf. Boole [20]). Euler diagrams are due to the Swiss mathematician Leonhard Euler, who described them in a series of letters he wrote in 1761 (Euler [42]).[39] They were mainly designed to the end of developing a diagrammatic version of syllogistic logic (cf. Shin [145], p.12).

Pierce's system of *alpha graphs* forms a diagrammatic version of sentential logic: every diagram in this system corresponds to a sentence in the language of sentential logic and *vice versa*. See Hammer [62], for a systematic study of this system.[40] Pierce himself also designed a system of so-called *beta graphs* and a system of so-called *gamma graphs*. The former can be considered a diagrammatic version of predicate logic, the latter of a kind of modal logic.[41]

Hammer [62], chapter 5, presents and studies a heterogeneous system. The language of this system includes Venn diagrams and sentences from first order logic. The computerized system Hyperproof (Barwise and Etchemendy [5]) may also be considered a heterogeneous system, as is explicitly done by Barwise and Hammer [10], p.88. Note that, strictly speaking, Hyperproof is not a logical system as characterized in § 2.1, since it involves the implementation on a computing machine.

According to Barwise and Etchemendy:

---

[39] See especially letters no. 102-8.

[40] But see Lemon [103], p.215, where doubt is expressed as to whether Pierce's system of alpha graphs may be genuinly called a diagrammatic system.

[41] Sowa's well-known work on conceptual graphs (Sowa [150]) is based on Pierce's system of alpha graphs and to some extent on Pierce's system of beta graphs.

> The importance of these results is this. They show that there is no principled distinction between inference formalisms that use text and those that use diagrams. One can have rigorous, logically sound (and complete) logical systems based on diagrams (Barwise and Etchemendy [6], p.214).

As indicated by this quotation, the study of proof and reasoning in terms of diagrammatic systems such as the ones mentioned above may be called *conservative* in certain respects (cf. Scotto di Luzio [142], p.118). The systems that have been developed form an attempt to accommodate for the apparent fact that reasoning may involve diagrams. In this respect, they certainly go beyond traditional sentential systems. Nevertheless, the techniques employed are entirely analogous to those in case of standard sentence-based systems. The members of the language are specified recursively, a model-theoretic semantics is provided, and hence a consequence relation. Logical proofs are again defined as certain sequences of members of the language. Subsequently one can try to prove these systems to be sound and complete. The study of diagrammatic and heterogeneous systems accordingly comes down doing "logic as usual."

This, however, is not an objection against the study of diagrammatic and heterogeneous systems *per se*. Rather, it puts such systems in what we deem to be their proper perspective. Two points may be raised.

First, defining proofs involving diagrams in terms of logical systems in a way as explained above puts diagrams in the same basket as sentences and accordingly treats them exactly alike. Specifically, just as sentences can follow from to other sentences, this likewise holds for diagrams. Thus, a diagram can follow from other diagrams and/or sentences. Alternatively, a sentence can follow from certain diagrams and/or sentences.

From a semantic point of view, this presupposes that diagrams have truth-values just as propositions do. It is accordingly that soundness (and completeness) of diagrammatic systems can be defined, and, if possible, proved. However, it is highly unnatural to hold that a diagram, as it occurs in a proof, should be understood as something that admits of a truth value. For example, it would seem somehow inappropriate to attribute a truth-value to, say, a diagram of a triangle, or a cube. Winterstein, Bundy, and Jamnik [162], p.194, express the point forcefully when they say that a "diagram cannot be true or false […]. We therefore do not define soundness in diagrammatic reasoning at all."

However, a qualification is in order. Many of the diagrammatic systems considered by logicians are often diagrammatic versions of well-known sentential systems (as was mentioned above). More generally, logicians have often been inclined to consider only those diagrams that can be quite naturally associated with sentences. Perhaps this points to a deeper reason why the diagrams considered by logicians are so naturally subjected to the usual logical methods. However, it is far from obvious how typical logical methods would apply to reasoning with diagrams that do not admit of truth-values. For example, it seems hard to see how the usual logical methods would take account of the proof described by Kant in The Passage. Indeed, the creation and modification of a diagram (a triangle in this case) and the inferences that are made accordingly appear not to accord very well with the usual logical frameworks.

A second point is that an account based on diagrammatic systems fails to take seriously the *spatial* aspects of a diagram (Lemon [103], Lemon and Pratt [104]). For example, one can use Euler diagrams in order to prove properties of various set theoretical relations and operations. However, in doing so, one crucially relies on various spatial relations of the curves that make up these diagrams. For example, in order to show that the inclusion relation among sets is transitive, it is precisely the *spatial* inclusion relation among the various curves exhibiting the set theoretical inclusion relation (see also § 2.3). The transitivity of the latter is shown precisely by means of the transitivity of the former. Accordingly, the spatial relations that in part constitute a diagram are really *employed* in the course of this type of reasoning. However, this point is swept entirely under the carpet by the logical approach to diagrammatic reasoning. For under this approach, a Venn diagram receives a set theoretical interpretation in the same way as sentences of well-known languages do. Accordingly, the spatial relations that in part constitute an Euler diagram are not accounted for.

In order to simplify our discussion, we shall henceforth restrict ourselves to sentential logical systems.

§ 2.2. Propositional reasoning

As regards the logical conception of proof, the reader will have noticed that language plays a quite crucial role. According to the logical conception of proof, a mathematical proof can be entirely formulated in terms of a language. Here it is to be understood that the sentence is the prime linguistic object—a language is a set of sentences. This points at a fundamental assumption underlying the logical conception of proof. The aim of this section is to articulate this assumption in various directions.

We proceed as follows. To begin with, we once more turn our attention to logical proofs and try to provide a deeper understanding of them. We point out that logical proofs can be (and in fact have been) understood in static as well as in dynamic ways. Next, we turn our attention to some of the tools used by logicians, and make clear how they have helped to shape the logical conception of proof. Finally, we turn to a development in logic that does not consider the proposition as the fundamental building block of a proof.

Logical proof texts. According to definition 1 above, a logical proof is a certain sequence of sentences. As it stands, a logical proof thus appears to be a static type of object, i.e., it is not subject to change through time. A logical proof has been more specifically thought of as (a model of) a certain type of text. The notion of text presupposed here is of a rather abstract sort: a text is simply a linear configuration of sentences.

Different types of texts have been distinguished in the literature, such as, for example, narrations and explanations. We may specifically refer to a logical proof as a *logical proof text*. See, for example, van Benthem [12], and Glymour [59], pp.14-5 for views along these lines. See also Vermeulen [161], who holds that a characteristic of a logical proof text is, among other things, the presence of certain inferential relations among the sentences constituting it.

It is clear that an important character figuring in logical proof texts is the sentence. This point manifests the central place of language within logical theorizing. Thus, in a recent edition of an introductory textbook we read on page 1: “Logic begins with the study of language” (Bonevac [19], p.1). The title of Barwise and Etchemendy’s *Language, proof and logic* [8] clearly suggests and intimate connection between logic, language, and—interestingly—proof.[42] Earlier, in 1936, we find the publication of Ayer’s well-known exposition on the doctrines of logical empiricism, titled *Language, truth and logic* [1]. Ayer gives us the impression of an intimate connection between logic and language (and truth). At the dawn of modern logic (in 1879), Frege’s influential *Begriffsschift, a formula language, modeled upon that of arithmetic, for pure thought* [47] was published. Considering the title, the logic set out in *Begriffsschrift* Frege appears to see as a certain type of *language* (a formula-language[43]), namely, a language for pure thought.

It may be considered remarkable that logic begins with the study of language outright, as for example Bonevac suggests (see above). Wouldn’t we expect that logic begins with the study of reasoning and proof? To this end, it may use whatever tools are appropriate in a given context. For example, given the question whether a given argument is valid, it seems natural to resort, to begin with, to an appropriate language in order to characterize the logical form of the sentences constituting the argument. However, when we turn our attention to the process structure of a proof, then it is far from obvious whether language forms the appropriate tool, though the matter may depend on the specific questions one seeks to answer. However, we think that also from this procedural point of view, the notion of language is, as it where, often built in our conception of proof: proof processes are often simply thought to be organized in terms of sentential items. Sentences are considered the prime medium in which reasoning takes place. Let us consider this issue somewhat more closely.

---

[42] This title may seem remarkable to some extent. For the software package delivered together with this book—we particularly have in mind the program Tarski’s world—allow a user to reason not only with sentences but also with diagrams.

[43] Frege’s original German term is *Formelsprache*.

Logical proof procedures. As said in the previous section, logical proof texts appear to be static in nature. Upon closer inspection, however, logical proofs can be, and often have been, thought of in dynamic terms as well. For example, a logical proof can be easily explained in process-oriented terms (as was noted by van Benthem [13]). In the light of this, reconsider also the definiens of definition 1 above, and in particular the phrase "is inferred by means of the application of a rule of natural deduction." This clearly suggests that a certain activity has taken place, namely, the application of a rule of inference. Consequently, it appears that definition 1 defines a logical proof in part as the product delivered by a certain process. This process can be understood as the production of a logical proof text. Related to this, logical proofs can be, and in fact have been, understood as cognitive procedures as well, namely, as cognitive procedure to the end of proving a theorem.

Let Π be a logical proof of ϕ. Let us think of Π as a logical proof text (cf. above). The sentence ϕ is the final sentence of this text. Now, we may imagine some agent, *a*, "interpreting and following" Π. Accordingly, after *a* has interpreted and followed Π, the proposition that ϕ is proved, i.e., its truth is established. Hence, *a* knows that ϕ.

This suggests us to think of a logical proof somewhat metaphorically as a kind of program expressing a certain procedure. In general, a procedure is a structured way of doing or acting. In the present context, a procedure is always carried out in order to reach a specific goal, namely, to establish the truth of a theorem. Let us refer to the procedure expressed by a logical proof as a *logical proof procedure*.

A logical proof procedure is a certain way of doing or acting in order to prove a theorem, that is, to prove that theorem *logically*. Bonevac has given us an impression of the way a logical proof procedure may proceed:[44]

> A mathematician may begin a proof by stating some assumptions. The mathematician then draws out consequences of the assumptions, perhaps making other assumptions along the way. Finally, the proof ends with a conclusion—the theorem it proves (Bonevac [19], p.2).

Cf. also Barwise and Etchemendy [8], pp.46-7; Frege [48], p.204.

In general, we can distinguish between a procedure on the one hand and a *run* of a procedure on the other. The latter we refer to as a *process*. A process is fundamentally characterized by its being in time. A procedure, in contrast, does not seem to be temporal, since it primarily concerns the *way* a process is to be carried out. We may say that the procedure provides control over the process: it determines what actions are executed and when. In this respect, then, the

[44] Reading *premises* where Bonevac uses *assumptions*.

procedure structures the process. In line with earlier terminology, we may refer to a run of a logical proof procedure as a *logical proof process* (but see below).

The post-condition obtaining after a run of a logical proof procedure is that the truth of a theorem is established, i.e., that a theorem is proved. We say that the proved theorem forms the *product* of a logical proof process. Suggestively put, we may say that by means of running a logical proof procedure, one *logically proves* a theorem, that, is a logical proof process proves a theorem in a logical way.

It may be wondered, however, how it is that a logical proof can express a logical proof procedure, and hence how a logical proof can structure a process delivering a theorem as its product. For after all, a logical proof is merely a certain sequence of sentences. A mere sequence of sentences does not in any way structure a process. There is no "flow of control," so to speak. In this respect, the program metaphor is perhaps somewhat misleading. The interpreter at least needs to know which sentences in the proof are axioms, which ones are assumptions, and which ones are inferred sentences. Also, he needs to know which sentences are used to infer other sentences from.

This difficulty can be met by defining the notion of logical proof in a somewhat wider sense. To begin with, note that when we normally write down a logical proof, we typically add consecutive line numbers to the respective sentences constituting that proof. Furthermore, we also typically "flag" the sentences in a logical proof with expressions such as 'axiom IV,' 'assumption,' '$\mathrm{I}\!\rightarrow\!(3,\ 6)$,' and '$\mathrm{E}\!\wedge\!(5)$.' These expressions indicate which sentences in the sequence are axioms, which ones are assumptions, and which sentences are inferred. Also, it is indicated which sentences are used in order to infer those sentences from. For example, we can read '$\mathrm{E}\!\wedge\!(5)$' as "infer this sentence from the sentence on line 5 by applying the elimination rule for the conjunction." As such, we may interpret these expressions as inference instructions. To some extent, we can attempt to provide control by building the line numbers and the inference instructions into a logical proof. The line numbers determine, so to speak, the order of execution; the inference instructions determine how to obtain a sentence is obtained and what other sentences are used in order to obtain it.

Collectively, this leads to a modification of definition 1 from § 2.1, as follows:

DEFINITION 1*. A *logical proof* is a finite sequence of consecutively numbered sentences and inference instructions. The inference instructions and line numbers are part of the proof. Every sentence in the sequence is an axiom, an assumption, or is inferred by means of a rule of natural deduction, using earlier sentences in the sequence.

(A similar proposal was made by Isard [83], p.295.)

However, it seems that this still cannot be the whole story. For example, what, precisely, should an interpreter do when he is interpreting an axiom, an assumption or any other sentence constituting a logical proof text? Given standard truth conditional semantics, the interpretation of a sentence yields the truth-value of that sentence. If the truth-value of a sentence is **true**, then that sentence is true; if not, not. However, it appears that truth-values on themselves do not in any way structure a process. Consequently, it is not clear how a truth-conditional account enables an interpreter to *follow* a logical proof procedure. Furthermore, this account is obviously problematic in case of assumptions and sentences dependent on assumptions. Assumptions, in particular, do not *have* a truth-value; they are merely assumed to have one.[45]

A way out of this difficulty is to interpret the sentences as instructions to act in certain ways.[46] Recall that any sentence of a logical proof belongs to either one of the following categories (cf. § 2.1):

1. axiom;
2. assumption;
3. sentence dependent on and assumption;
4. theorem.

The axioms and the theorems express propositional items of knowledge. The assumptions and sentences dependent on assumptions, in contrast, express mere propositions (cf. above).

Philosophers have suggested that sentences may be interpreted in terms of acts of judgment, e.g., as an act of assenting to the truth of a proposition (cf. Dummett [37], p.362, Sundholm [153]; Frege [47] is an earlier defender of such a view). We can further this proposal by distinguishing between immediate judgments (axioms) and mediate judgments (theorems). Alternatively, one may interpret axioms and theorems in terms of a corresponding linguistic act of assertion. Assumptions (i.e., sentences flagged with 'assumption'), furthermore, may be interpreted as acts of taking a certain sentence to be true, if only temporary. Sentences dependent on an assumption may be taken as acts of conditional judgment or assertion, that is, as procedures of judging or asserting something to be true on the condition or assertion that some other proposition is true (Belnap [11]).[47]

---

[45] Accordingly, standard truth-conditional semantics is not appropriate for assumptions and sentences dependent on assumptions. See Belnap [11].
[46] Such a line of thought motivates much work in dynamic semantics; see van Benthem [13] for a discussion and overview of the field; see also Muskens, van Benthem, Visser [113].
[47] Sundholm [154] prefers a categorical analysis of judgment and assertion, and accordingly takes a different route.

In sum, the main types of procedure carried out in the course of a logical proof procedure are acts of judgment, acts of assumption, acts of conditional judgment, and acts of inference.

After a logical proof procedure has been executed, what is left is the product delivered by it, i.e., the proved theorem. The logical proof process itself need not exist anymore. Perhaps this is what Hardy and Littlewood had in mind when they compared a proof with a "gas". Their idea seems to be that a proof "evaporates" as soon as the theorem is proved (cf. Hardy [63], p.18). However, a logical proof process can be "traced" by recording the axioms, assumptions and the other propositions that have been used, for example, by writing them down. Accordingly, one again obtains a logical proof, but this time understood as a logical proof text. As seen accordingly, a logical proof text in some sense "fixates" a logical proof process. We may refer to this text as the *result* of a logical proof process.

In sum, we have seen that we can think of logical proofs from a more dynamic as well as a more static point of view. Despite the fact that these two points of view can certainly be distinguished, we do not think that they can be separated. Thus, for instance, while a logical proof text is certainly different from the logical proof procedure encoded by it, both often manifest a delicate and subtle interplay. This is witnessed by the fact that we so easily explain to others the result (a text) in terms of a procedure that delivers it (see above). In the other direction, during a run of a logical proof procedure, it is often required to check whether the product delivered thus far satisfies certain conditions. For example, in order for a rule of inference to be correctly applied one often needs to verify whether certain conditions obtain. To this end, one often looks back at the static presentation of the proof so far, which is often formulated in terms of a text.

Taken together, we think that a logical proof—indeed, a proof generally—has a multiplicity of dimensions. By way of analogy, we may compare a logical proof with a vector in a high dimensional space. So conceived, logical proofs conceived of as texts and logical proofs conceived of as procedures correspond to two different "projections" (or aspects) of the same rather than two separate kinds of things. Though static and dynamic aspects of logical proofs can certainly be distinguished, it is hard, if not impossible, to split them apart.

Something along these lines seems to be suggested by a certain ambiguity that can be found in the term *proof*, and hence *logical proof* (cf. Sundholm [153]). On the one hand, the term may refer to a certain dynamic procedure—e.g., a logical proof procedure. On the other hand, it may refer to the result of such a procedure. The single word *proof* in a sense unites both the dynamic and the static aspects we have pointed out above.

As the considerations given above should have made clear, we will not succeed in concentrating *exclusively* on the dynamic dimensions of logical proofs. Nevertheless, the reader will notice that in what follows we tend to think of logical proofs primarily as logical proof procedures. The reason is closely connected to the fact that in the present study we are primarily interested in the cognitive aspects of proofs. Particularly in the present chapter, we want to obtain more insight in *how* a theorem is logically proved. Naturally enough, then, procedural considerations will come to the fore.

Language. For several times we have stressed the central place of language in the logical conception of proof. Let us brief look somewhat more closely to the languages used by logicians. We shall mainly consider the syntax of so-called sentential languages and first-order languages. Though other languages have been considered by logicians, we can make our points only on the basis of these. For our purposes, the syntax of these languages is not very interesting on itself. Nevertheless, considering their syntactic specification allows us to put them in an interesting perspective. We begin with the syntax of sentential languages.

1. *Sentential languages*. A sentence of a sentential language can be seen as being built from two main types of (syntactically) *primitive expression*: primitive sentences and logical constants. Thus, let P be a set of primitive sentences. Specific logical constants often considered are '$\wedge$' (conjunction), '$\vee$' (disjunction), '$\rightarrow$' (material conditional), and '$\neg$' (negation).[48] We also add parentheses (e.g., '(' and ')') to the primitive expressions. They mainly serve to avoid syntactic ambiguity.[49]

From a syntactical point of view, a sentential language $\mathscr{L}$ consists of certain strings of primitive expressions that we wish to consider as well formed. Thus, define an *expression* as any string of primitive expressions. Let A be a set of expressions that satisfies the following two conditions:

(i) any primitive sentence in P is in A;
(ii) if ϕ and ψ are in A, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $\neg(\phi)$ are in A.

Then the sentential language $\mathscr{L}$ is defined as the intersection of all sets A satisfying (i) and (ii). As usual, outermost parentheses are often omitted; whenever ϕ is a primitive sentence, $\neg(\phi)$ is often written as $\neg\phi$.[50]

As an example, suppose the following primitive sentences are in P: 2 *is prime*, $7 + 5 = 12$, 7 *is a square number*. Then the following are examples of other (compound) sentences of $\mathscr{L}$:

[48] Cf. also footnote 29.
[49] Representing the syntax of sentences in an alternative way (e.g., by using Polish notation) allows one to dispense with parentheses. Also, some logical constants can be defined in terms of others, allowing one to dispense with the former. We shall not consider these issues here.
[50] Thus, it is understood that negation dominates over all other logical constants.

- 2 *is prime* $\wedge\ 7 + 5 = 12$,
- $\neg 7$ *is a square number*,
- (7 *is a square number* $\vee\ 2$ *is prime*) $\rightarrow 7 + 5 = 12$.

2. *First-order languages*. A first general difference between a first-order language and a sentential language is that sentences of a first-order language are built from two extra logical constants: $\forall$ (universal quantifier) and $\exists$ (existential quantifier).

A second difference is that, instead of primitive sentences, other types of non-logical primitive expressions are considered. These are: relational expressions, functional expressions, individual constants and variables.

To any relational expression and to every functional expression there corresponds an integer ≥ 1, called the *arity* of that relational or functional expression. (A relational expression of arity 1 is called a *predicate*.) The arity of a relational expression determines the number of relata that can be associated with that expression; the arity of a functional expression determines the number of arguments that can be associated with that expression.[51]

We shall not give an official definition of a first-order language here (see, for example, Barwise and Etchemendy [8], pp.231-2). Instead, we give the reader a feel for what sentences in a first-order language look like by way of a few examples.

Consider the predicate *is prime*. Also, consider the following relational expressions: $=$ (the identity relation), and $>$ ("greater than"), both of arity 2.[52] Furthermore, suppose we are given functional expression $+$ (addition) and $\cdot$ (multiplication), both of arity 2. Finally, suppose that also the individual constants 1 (one) and 2 (two), 5 (five), 7 (seven), and 12 (twelve). Then the following are examples of sentences of a first-order language:

- $\forall x(x$ *is prime* $\rightarrow \exists y(\, y$ *is prime* $x \;\wedge\; y > x))$,
- $\forall x \exists y(x \cdot\; y = 1 \;\wedge\; y \cdot x \;= 1)$,
- $7 + 5 = 12$,
- 2 *is prime*.

Let us compare sentential languages with first order languages.

To begin with, notice that while sentences such as 2 *is prime* and $7 + 5 = 12$ are considered as primitive in a sentential language, they are considered composite in a first-order language. A similar remark can be made on behalf the

[51] In some presentations, functional expressions of arity 0 are also allowed. Accordingly, constants are assimilated under functional expressions, namely, as functional expressions of arity 0 (e.g, Monk [110]).
[52] We do not consider '=' a logical constant.

first and the second example. Both sentences are considered primitive from the point of view of a sentential language. In contrast, they are considered composite from the point of view of a first-order language.

A similar point can be made with respect to other types of languages that we have not considered in any detail. To illustrate this, consider the sentence $\Box$(2 *is prime*).[53] From the point of view of a sentential language, this sentence is considered primitive. Suppose, now, that we consider the $\Box$ as a logical constant. Thus, we arrive at a *modal sentential* language. From the point of view of this language, the aforementioned sentence has two primitive constituents, namely, the logical constant $\Box$ and the (primitive) sentence 2 *is prime*.

Similarly, if we consider the sentence $\Box$(2 *is prime*) from the point of view of a standard first-order language as considered above, then this sentence has two primitive constituents, namely, the constant 2 and the predicate $\Box$ … *is prime* (arity 1). Again, suppose we now consider $\Box$ as a logical constant, as we do from the point of view of a modal first-order language. Then this sentence has three primitive constituents, namely, the logical constant $\Box$, the constant 2, and the relational expression *is prime*. Let us state our conclusions.

In some sense, sentential languages are the main languages among those considered by logicians. With respect to sentential languages, other languages add more syntactic fine structure at the sentence level (though often in incomparable ways). Related to this, sentential logic (or propositional logic) can be seen as the "mother logic." In a way, predicate logic, modal logic and many other logics are all sentential logics. We do not mean sentential logics in the technical sense, i.e., as a specific type of logical system. The point is that sentential logic is *a logic of sentences* (or a logic of propositions). Accordingly, reasoning within sentential logic is reasoning with sentences. Other logical systems such as predicate logic, modal logic, and so on, inherit precisely this property.[54]

Recall our earlier point that modern logic assumes that mathematical proofs are at the fundamental level built from propositional items. Our point now is that this assumption is intimately related to the nature of (some of) the tools used by logicians in order to study mathematical proofs. In this respect, the most important tools we have in mind are languages. There is an intimate relation between the structural organization of a logical proof and language (sentences). Put differently, the ubiquitous use of languages by logicians betrays a very

[53] To be read as: necessarily, 2 is prime.

[54] By way of qualification, let us note that one may consider systems based on the sequent calculus as taking a short step away from this idea. For in case of the sequent calculus, a proof at the fundamental level consists not of just sentences but of *sequents*. A sequent is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are sets (or sequences, or multisets) of sentences in the usual sense. A similar remark applies to systems based on labeled deduction. In that case, the basic unit of a proof is not just a sentence but a *labeled sentence*. A labeled sentence is an expression of the form $\phi : x$, where ϕ is a sentence in the usual sense and x is a term of a given labeling language. See also the next section on term logic.

specific view on the structural organization of a mathematical proof. This point holds independently of the fact whether one thinks of proof primarily as texts or as certain procedures (see above). Consequently, to put into question the assumption that mathematical proofs are at the fundamental level built from sentences will inevitably lead to a critical consideration of these tools and *vice versa*.

The very practice of studying proofs through their representation in a language betrays a fundamental assumption determining our conception of what a mathematical proof is, and in particular, what the fundamental building blocks of a mathematical proof are. Perhaps this is one of the things van Heijenoort had in mind when he said that the founding fathers of modern logic—van Heijenoort mentions people like Frege and Russell in particular—thought of "logic as a language" (cf. van Heijenoort [65]).[55] One idea van Heijnenoort tries to point out is that according to this conception of logic, reasoning is quite literally reasoning *within* a logical system: proving a theorem, for example, means primarily *using* the axioms, rules, and other sentences of that system in order to prove that theorem (cf. *ibid*., p.326).[56]

Accordingly, language—or, more precisely, sentences—forms the prime medium of reasoning. It is precisely this point constituting a fundamental assumption underlying the logical conception of proof. In chapter 3 we shall see that Kant did not accept this assumption. Accordingly, an interpretation of Kant's views on proof in terms of the logical conception of proof misreads Kant on a fundamental point. This, however, is of later concern.

Term logic. It is of interest to notice that even from the standpoint of modern logic, scholars have considered alternative formats of the building blocks of proof.[57] However, it should be admitted that these developments have remained somewhat off mainstream.

Thus, Fred Sommers, in collaboration with George Englebretsen, has developed a logic of terms, or *term logic* (abbreviated as TL). We shall not enter into the details of the system considered by Sommers and Englebretsen. Instead, we only highlight a few salient points. The interested reader is referred to Sommers and Englebretsen [149] for a much more extensive and detailed exposition.

Historically speaking, TL has its roots in Aristotelian syllogistic logic. A negative point often brought up against syllogistic logic is its lack of expressiveness: within syllogistic logic, one cannot reason with relations—

---

[55] See also Hintikka [78].

[56] Let us note that Frege would deny that his logic is a logic of *propositions*. In contrast, the logical system presented in *Begriffsschrift* [47] is strictly speaking a logic of *judgments*. Despite this qualification, our main point nevertheless remains standing: for Frege, too, the structural organization of a proof is intimately related to language (sentences).

[57] See also § 3.1.

relational *expressions*, that is. The logic of terms is intended to accommodate for this lack. In a sense, TL incorporates first-order logic. However, TL takes a somewhat different orientation towards the internal organization of reasoning (see also § 3.1).

Let us provide some background. Syllogistic logic can be seen as a logic of four types of *categorical sentence*:[58]

(a) *all A are B*
(i) *some A are B*
(e) *all A are not B*
(o) *some A are not B*.

In case of (a), (i), (e) and (o), *A* is called the *subject term* and *B* is called the *predicate term*.

Let us call *not*, as it occurs in both (e) and (o), a *sign of quality*. Alternatively, *not* may also occur as a *sign of judgment*, as in *not some A are B*. The latter may also be written as *no A are B*, which is closer to common English usage (this sentence is equivalent to (e) above).

It is easy to see that, for instance, the sentence *every boy likes a girl* cannot be very well expressed within syllogistic logic. The main reason is that the sentence involves the relational expression *likes*, which cannot be accommodated for. Within a first-order language, in contrast, this sentence can be paraphrased as:

$$\forall x(x \textit{ is a boy} \rightarrow \exists y(y \textit{ is a girl} \wedge x \textit{ likes } y)).$$

Using restricted quantifiers, this sentence might be construed as follows:

$$\forall(x : x \textit{ is a boy})(\exists(y : y \textit{ is a girl})\ x \textit{ likes } y),$$

or, employing indices instead of variables:

$$\forall\ \textit{boy}_1(\exists\ \textit{girl}_2\ \textit{likes}_{12}).$$

Within TL, *all* (or $\forall$) is written as '–' and *some* (or $\exists$) is written as '+.' Furthermore, *not* (either as sign of quality or sign of judgment) is written as '–' and *are* is written as '+.'

Observe that these notations suggest an analogy with arithmetic, which is indeed what Sommers and Englebretsen have in mind. Note, however, that both '–' and '+' are overloaded. While this facilitates reasoning as a kind of algebraic

[58] But see footnote 62.

calculation (see below), it also requires certain restrictions, breaking down a full analogy with arithmetic. For example, within TL we have

$$-(-A + B) = +A + (-B),$$

but not

$$-(-A) + B = +A + B.$$[59]

The latter runs contrary to properties of the corresponding arithmetic operations of addition and subtraction. Let us return to our original discussion.

With the help of the notations introduced above, our last example can be rewritten as:

$$-boy_1 + (girl_2 + likes_{12}).$$

This algebraic expression is in the "language" of TL. Specifically, the expression may be seen as a complex term, built from the primitive terms boy_1, $girl_2$ and $likes_{12}$, and the algebraic operations − and +.

Now, consider also the sentence *every girl owns a dog*. Within TL, this sentence can be paraphrased as

$$-girl_2 + (owns_{23} + dog_3).$$

The sentences *every boy likes a girl* and *every girl owns a dog* imply *every boy likes the owner of some dog*. Interestingly, within TL, this can be shown by way of algebraic calculation:

$$[-boy_1 + (girl_2 + likes_{12})] + [-girl_2 + (owns_{23} + dog_3)]$$

$$= -boy_1 + (likes_{12} + (owns_{23} + dog_3)).$$[60]

In view of this example, we can make the following point.

Given an appropriate specification of the rules for the operations + and − (Sommers and Englebretsen develop these in their book; see *ibid.*), reasoning within TL generally comes down to a kind of algebraic calculation with terms.[61]

---

[59] This may be respectively read as: *not all A are B* is equivalent with *some A are not B*, while *all not A are B* is not equivalent with *some A are B*.

[60] Because *girl* in the second premise is universal, it may be assigned any subscript. (Let us parenthetically add that the addition of subscripts generally is non-deterministic, and hence this may affect the outcome of the calculations.)

[61] It turns out that TL in some sense includes propositional logic. For example, the term $A + B$ can be shown to behave like the conjunctive "*A and B*." It will now be expected (apply DeMorgan's

Thus, in comparison with logics of sentences, TL involves an interesting shift as to the format of the items reasoned with in the logic. From the point of view of logics of the former type, reasoning is at the fundamental level seen as reasoning with sentences. Reasoning within TL, in contrast, may in general be thought of as a kind of reasoning with *terms*.[62]

It appears, however, that logics such as TL have thus far not been a very influential means to think about mathematical proof. As yet, they have not been incorporated within the mainstream. Borrowing from Kuhn, we may say that nowadays "logics of sentences" in general still constitute the dominant paradigm.

§ 2.3. Two types of logical inference

In the previous section, we have seen that the logical conception of proof assumes that sentences form the prime medium of reasoning.[63] We argue in chapter 3 that Kant did not accept this assumption, a point that will be developed in a positive way in chapters 4 and 5. Nevertheless, many people have found that the logical conception of proof can be of help in understanding Kant. Hintikka forms an important example in this respect (see above; cf. also below). We disagree, and for reasons indicated earlier: Kant does not accept that sentences are the prime medium of reasoning.

Nevertheless, we think there is an element of the logical conception of proof that has some similarities with a fundamental notion figuring in Kant's conception of mathematical proof. This notion can be put under the heading of *construction*. The similarity has to be taken with care, however, and we provide the necessary qualifications in due course. Though we think that logical proofs form in the end an inadequate means to reconstruct Kantian mathematical proofs (let us call them), the present section provides a small qualification to this standpoint. Thus, the aim of the present section is to bring out a certain constructive element of logical proofs. We proceed as follows.

Relative to systems of natural deduction, two types of logical inference can be distinguished. We respectively call them: *simple logical inference* and *annotated logical inference*. Both types of logical inference will be successively introduced and discussed below. It is annotated logical inference that betrays the constructive feature that we wish to bring out. Next, we link our conclusions to themes to be raised in chapter 4.

---

law) that $-[(-A) + (-B)]$ behaves like "*A or B*," which indeed turns out to be the case. Furthermore, $-A + B$ can be shown to behave like "*if A then B*." Thus, in a way, TL incorporates propositional reasoning as well. (Note that '+' now also stands for propositional conjunction.)

[62] Incidentally, note that from the point of view of TL, syllogistic logic does not appear as a logic of categorical *propositions* but as a logic of *terms* instead.

[63] Term logic formed an exception in this respect.

Simple logical inference. We begin by briefly considering inference generally. Next, we narrow our focus to logical inference, and simple logical inference in particular. Terminology will be clarified as we go along.

In the present study, we mainly wish to think of inferences in procedural terms. Let us provisionally say that an *inferential procedure* (or an *inference* for short) is a procedure to the end of drawing a conclusion. A run of an inferential procedure is called an *inferential process*. The product of an inferential process is what we call a *conclusion*. We assume that a conclusion is always a proposition.[64] Instead of "to infer ϕ", we will alternatively say: "to conclude ϕ."

An important characteristic of *logical* inference is its highly non-local nature. This point has also been expressed by saying that logical inference is *topic neutral* (Detlefsen [33]). The latter has been taken to mean, among other things, that a logical inference can be made irrespective of the specific subject-matter one is reasoning about. As an example, consider a logical inference in accordance with *modus ponens* (see also below). It allows us to infer ψ from $\phi \rightarrow \psi$ and ϕ. The idea now is that one is allowed to carry out this inference no matter whether ϕ and ψ refer to physical objects, mathematical objects, or any other type of object. In a sense, the inference is indifferent to its subject-matter.

As opposed to logical inference, one may imagine the possibility of a type of inference that is more intimately related to the specific subject-matter one is reasoning about. In this respect, such a type of inference would be more local in nature in comparison with logical inference. For example, Poincaré [123], p.37, believed that an inference in accordance with the principle of mathematical induction (or complete induction) is such a type of inference. According to the principle of mathematical induction, one is allowed to conclude that all natural numbers have a certain property P (say) if one knows that (1) 0 has property P and (2) for all natural numbers n, $n + 1$ has P provided that n has it.

In Poincaré's view, an inference in accordance with the principle of mathematical induction is a distinctly *mathematical* type of inference. As such, it is intimately related to a specific mathematical subject-matter one is reasoning about. More specifically, in Poincaré's view, we may say that an inference in accordance with the principle of mathematical induction is a type of inference that is intimately related to reasoning about natural *numbers* (cf. *ibid*., pp.37-9). Let us return to our original discussion.

A *simple logical inference* is a logical inference such that the conclusion is directly drawn from one or more other propositions. A simple logical inference can be best seen as an inferential transition from one or more propositions to another. The propositions from which the conclusion is drawn are supposed to be given. An *annotated logical inference* is an inference such that a conclusion is

[64] Recall from § 2.1 that logical systems have been designed allowing for inferences where a diagram is drawn as a conclusion. (We also said that, in this study, diagrammatic logical systems will not be considered.)

not dawn from one or more given propositions. In contrast, it is an essential characteristic of annotated logical inferences that a certain derivation be produced first. A derivation is a sequence of sentences that is in many respects like a proof, except for the fact that it does not proceed solely from axioms but also from assumptions. How this derivation is to be produced is circumscribed by a certain restriction, or "annotation," and hence *annotated* logical inference. In a way, the conclusion is then drawn from this derivation, or from the production of this derivation (see below).

A simple logical inference is carried out in accordance with what may call a *simple rule of inference* (or a *simple rule of natural deduction*). An annotated logical inference is carried out in accordance with what we call an *annotated rule of inference* (or an *annotated rule of natural deduction*).

Unless confusion is possible, we shall henceforth omit the adjective 'logical' and speak in terms of *simple* and *annotated* inference respectively. A similar point holds for the corresponding rules of inference. Before we illustrate these two types of inference, let us first explain why it is so interesting to consider them.

Simple inferences correspond to what Prawitz calls *proper inferences*; annotated inferences correspond to what Prawitz calls *improper inferences*. Improper inferences (i.e., annotated inferences) are those in accordance with the introduction rule for the material conditional, the introduction rule for the universal quantifier, and the elimination of the existential quantifier. Proper inferences (i.e., simple inferences) are inferences respectively in accordance with the introduction or elimination rule for the conjunction, the introduction or elimination for the disjunction; the introduction and elimination rule for the negation, the elimination rule for the material conditional, the elimination rule for the universal quantifier, and the introduction rule for the existential quantifier.[65] (See below for illustration and discussion of the rules mentioned.)

Our reason for diverging from Prawitz' terminology is as follows. The term *improper inference* seems to suggest that improper inferences, in contrast with proper inferences, are somehow not "proper" or genuine inferences. However, we wish to avoid any suggestion along these lines. In our view, there is no reason to consider annotated inferences as somehow "improper." We think it is more fruitful to admit a variety of types of logical inference and to consider them all as genuine (or "proper"). Let us now turn to a discussion of simple inferences.

Where $\phi_1,\ldots, \phi_n$ ($n \geq 1$) and ψ are sentences, we can write a *simple* (*logical*) *inference* as a configuration of the following form:

$$\frac{\phi_1,\ldots, \phi_n}{\psi.}$$

[65] Prawitz uses a different but "equivalent" collection of rules (cf. *ibid*., p.20). However, our division of proper and improper inferences exactly follows his intentions.

The sentences $\phi_1,\ldots, \phi_n$ are called the *premises* of the inference; ψ is called the *conclusion* of the inference.

A possible premise of a simple inference is allowed to be an axiom, an assumption, a sentence dependent on an assumption, or a theorem. Note, however, that the term *premise* is now overloaded. Indeed, in § 2.1, we referred to premises as certain sentences cited in a logical proof (§ 2.1, definition 1). Later, however, we referred to these sentences more specifically as axioms (cf. § 2.1, definition 1). Accordingly, the ambiguity noticed is not very harmful.

In line with our more general characterization of inference above, we think of a configuration as the one above as encoding a certain procedure. Somewhat more specifically, we think of such a configuration as encoding a procedure of drawing ψ as a conclusion from $\phi_1,\ldots, \phi_n$ together. We may refer to such a procedure as a *simple inferential procedure*. However, we shall often continue to speak in terms of *simple inferences*, except on occasions where we want to stress the procedural dimensions of such inferences. A run of a simple inferential procedure can be called a *simple inferential process*. The product delivered by such a process is precisely the conclusion drawn. We sometimes say that the conclusion is drawn *from* the premises.

In a way, simple inference forms the stereotypical type of inference. A simple inferential procedure can be seen as an inferential transition from one or more premises to a conclusion. It seems to us that something along these lines is what many people mainly have in mind when thinking of inference. However, it turns out that inference is much more varied than this, even within the context of logical systems. Our distinction between simple inference and annotated inference shows this. Furthermore, and seemingly outside the context of well-known logical systems, still other types of inference can be found (see below; see also § 4.2).

A logical inference refers to a *rule* of inference. As said earlier (§ 2.1), a logical rule of inference generally can be seen as a license to infer a conclusion using other sentences.

A rule of inference of a system of natural deduction is called a rule of natural deduction (§ 2.1). Every rule of natural deduction is an introduction rule or an elimination rule for a specific logical constant.[66] An introduction rule can be seen as introducing a logical constant in the conclusion (as the principal constant). An elimination rule can be seen as eliminating a principal logical constant from one of the premises. In this respect, reasoning within a system of natural deduction can be seen as a succession of eliminations and introductions of logical constants. Accordingly, logical constants play quite literally a pivotal role when it comes to reasoning within a system of natural deduction.

[66] This is already clear by now in case of the logical constants $\vee$, $\wedge$, $\exists$, and $\forall$. The introduction rules for respectively $\rightarrow$ and $\neg$ will be presented in the next section.

Corresponding to the distinction between simple and annotated inferences, there are simple and annotated *rules* of natural deduction respectively. In the present section, we only consider simple rules of natural deduction.

An example of well-known simple rule of natural deduction is *modus ponens* (we mentioned this rule earlier).[67] According to this rule, one is licensed to infer a conclusion ψ from two premises ϕ and $\phi \rightarrow \psi$. An inference in accordance with *modus ponens* can be written as follows:

$$\frac{\phi, \phi \rightarrow \psi}{\psi.}$$

The inference written above *is* not the rule *modus ponens*. In contrast, it represents an inference in accordance with this rule. The difference can be marked out by representing rules of inference as *inferential schemas*, or *schemas*, for short. *Modus ponens* itself can be written as the following schema:

$$\frac{\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B}}{\mathbf{B}.}$$

Here, **A** and **B** are schematic letters standing in place of sentences.

Generally speaking, any simple rule of natural deduction can be represented in terms of the following *inferential schema* (or *schema* for short):

$$\frac{\mathbf{A}_1,\ldots, \mathbf{A}_n}{\mathbf{B}.}$$

Here, $\mathbf{A}_1,\ldots, \mathbf{A}_n$ are schematic letters stand in place of the respective premises of a simple inference; **B** is a schematic letter that stands in place of the conclusion of that inference. In what follows, however, we are not so much interested in inferential schema's as well as the inferences that are carried out in accordance with the rules of natural deduction represented by such schema's.

Another example of a simple rule of natural deduction is known as the *elimination rule for the universal quantifier*. An inference in accordance with this rule may be written as follows:

$$\frac{\forall x\phi(x)}{\phi(a).}$$

In words: infer that the object a is ϕ given that everything is ϕ.[68]

[67] Also known as the *introduction rule for the material conditional.*
[68] Note that ϕ, as it occurs in the premise of this inference, is not a sentence but an *open sentence*, i.e., a sentence in which at least one free variable occurs.

Let us mention the remaining simple rules of natural deduction. Consider the following inferences:

$$\frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi, \psi}{\phi \wedge \psi} \qquad \frac{\phi}{\phi \vee \psi}$$

$$\frac{\phi \vee \psi, \phi \to \chi, \psi \to \chi}{\chi} \qquad \frac{\phi(a)}{\exists x \phi(x)} \qquad \frac{\neg\neg\phi}{\phi.}$$

The first inference on the first row is in accordance with the *elimination rule for the conjunction*.[69] The second inference is in accordance with the introduction rule for the conjunction.[70] The third inference on the second row is in accordance with the introduction rule for the disjunction. The first inference on the second row is in accordance with the elimination rule for the disjunction. The second is in accordance with the introduction rule for the existential quantifier. Finally, the third inference on the second row is in accordance the elimination rule for double negations.[71]

Annotated logical inference. As said, in case of annotated inferences, a conclusion is not drawn via an inferential transition from one or more given premises. In contrast, it is an essential characteristic of annotated inferences that a derivation be produced first. In a way, the conclusion is then drawn from this derivation, or from the production of this derivation. In this section, we consider the annotated rules of natural deduction. These are: the introduction rule for the material conditional, the introduction rule for the universal quantifier, the introduction rule for the negation and the elimination rule for the existential quantifier.

Let us begin by considering the introduction rule for the material conditional. Where ϕ and ψ are sentences, an inference of this type is often written as follows (cf., e.g., Prawitz [128], p.20):

---

[69] Strictly speaking, there are two elimination rules for the conjunction. According to the rule mentioned in the text, one is licensed to "infer to the right conjunct." According to the other, one is licensed to "infer to the left conjunct," that is, on is licensed to infer ψ from $\phi \wedge \psi$.

[70] Strictly speaking, there are two introduction rules for the disjunction. According to the rule mentioned in the text, one is licensed to "infer a disjunct to the right." According to the other, one is licensed to "infer a disjunct to the left," that is, on is licensed to infer $\phi \vee \psi$ from ψ.

[71] This rule is not admissible within systems of natural deduction for intuitionistic logic. In contrast, it is admissible in systems of natural deduction for classical logic. It is also convenient to have the "repetition rule" available (both in an intuitionistic as well as in a classical setting). According to this rule, one is licensed to infer ϕ from ϕ. We will not consider this rule here.

$$
\begin{array}{l}
\ulcorner \phi \\
\quad \vdots \\
\quad \psi \\
\hline
\phi \rightarrow \psi.
\end{array}
$$

The idea behind this inference is as follows. In order to infer $\phi \rightarrow \psi$ (say), one begins by assuming ϕ, that is, the antecedent of the conditional. Subsequently, one infers the consequent of the conditional, ψ, from this assumption together with the remaining active assumptions (see below) and the axioms cited in the proof.

However, the above configuration does not characterize the introduction rule for the implication completely. There is a restriction circumscribing this rule: one is allowed to infer $\phi \rightarrow \psi$ if at this point there are no other active assumptions that where introduced after ϕ was assumed. If that is the case, we say that the assumption ϕ (including the sentences that have been inferred from it) is *discharged*. (That ϕ has been discharged is indicated by the "hook".) If an assumption has still not been discharged, we call that assumption *active*.

The above inference is not a simple logical inference: it is not an inferential transition from one or more premises to the conclusion $\phi \rightarrow \psi$. What else is it that this conclusion is drawn *from*? Prawitz has made an interesting proposal for an answer. He suggests us that the "premise" from which $\phi \rightarrow \psi$ is inferred is a *derivation* (see below) of ψ from the assumption ϕ together with the assumptions active at the point $\phi \rightarrow \psi$ is inferred and the axioms cited in the proof (cf. Prawitz [128], p.23).

A *derivation* of a sentence ψ is not a logical proof of ψ. The point is that there possibly still are active assumptions on which ψ depends. This is not allowed in case of a logical proof of ψ (see § 2.1). As such, the truth of ψ is not established. In contrast, it is merely established that ψ is true *on the condition* that the active assumptions are true, hence *derivation*.

With Prawitz' proposal in mind, it becomes perhaps more suggestive to write the inference we are currently considering as follows ($\alpha_1, \ldots, \alpha_n$ are the axioms cited in the proof; $\beta_1, \ldots, \beta_m$ are the active assumptions at the point $\phi \rightarrow \psi$ is inferred:

$$
\begin{array}{rl}
 & \alpha_1 \\
 & \vdots \\
 & \alpha_n \\
 & \vdots \\
 & \beta_1 \\
 & \vdots \\
 & \beta_m \\
 & \vdots \\
\text{assumption} \longrightarrow & \phi \\
 & \vdots \\
 & \psi \\
\hline
 & \phi \rightarrow \psi.
\end{array}
$$

Thus, the idea is that $\phi \rightarrow \psi$ is inferred from the entire derivation

$$\alpha_1,\ldots, \alpha_n,\ldots, \beta_1,\ldots, \beta_m,\ldots, \phi,\ldots, \psi$$

From the point of view of a system of natural deduction, a derivation is something that must be *produced* from other propositions. In order to produce a derivation of ψ from ϕ one will typically carry out several intermediate inferences, yielding several intermediate conclusions along the way. However, while it accordingly clear enough where these intermediate conclusions come from, the following will perhaps come as unexpected: where does the assumption ϕ come from? Thus far, it simply appears to fall out of the sky.

As a first shot, one may propose that this assumption is taken from the language of the underlying system. However, this would seem to make the very possibility of producing a derivation of ψ from ϕ dependent on available resources: what if ϕ is not a sentence of this language? A way out may be to suggest that one *creates* the assumption ϕ. However, it seems that we have now arrived at the darker realms of modern logic. We lay this matter to rest.

The next example of a constructive inference is known as the introduction rule for the universal quantifier. According to this rule, one is licensed to infer $\forall x\phi(x)$ given $\phi(a)$. In other words, one is allowed to infer that everything is ϕ given that a is ϕ. However, the configuration below does not completely characterize the introduction rule for the universal quantifier:

$$\frac{\phi(a)}{\forall x\phi(x).}$$

Indeed, given that an object a satisfies $\phi(x)$, it does not follow that every object satisfies $\phi(x)$.[72]

The point can be remedied by invoking a restriction circumscribing this rule: in order to infer $\forall x\phi(x)$ from $\phi(a)$, the constant a must not occur in an axiom, or in an active assumption, or the sentence $\forall x\phi(x)$.[73] Accordingly, the above configuration is circumscribed by this restriction.

In a sense, this restriction on possible applications of the introduction rule for the universal quantifier secures that a is "arbitrary." The idea is that, instead of a, one might as well have used any other constant. This can be secured if, in order to infer $\forall x\phi(x)$, no specific knowledge bearing on a is used at the point one infers $\forall x\phi(x)$, except for $\phi(a)$ itself. It is essentially this what the restriction tries to capture.

Accordingly, it appears that $\forall x\phi(x)$ is not simply inferred from $\phi(a)$, but rather from a specific derivation of $\phi(a)$ from axioms and (active) assumptions. Again, denoting these axioms as $\alpha_1,\ldots, \alpha_n$ and the assumptions as $\beta_1,\ldots, \beta_m$, we can suggestively write the introduction rule for the universal quantifier as follows:

$$\frac{\boxed{\begin{array}{l}\alpha_1\\ \vdots\\ \alpha_n\\ \vdots\\ \beta_1\\ \vdots\\ \beta_m\\ \vdots\\ \phi(a)\end{array}}}{\forall x\phi(x).}$$

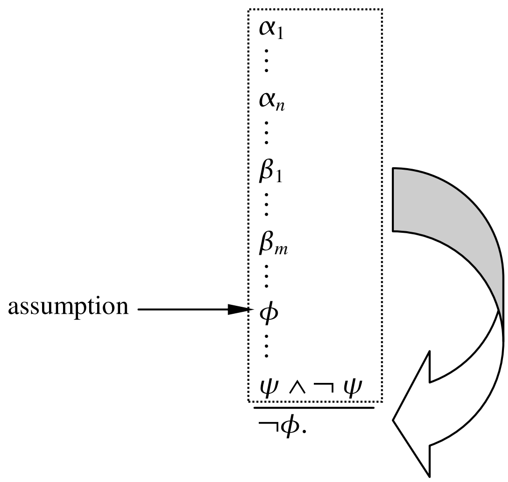
Let us present the remaining annotated rules of inference.

The first is the introduction rule for the negation. The idea underlying an inference in accordance with this rule is as follows. In order to infer $\neg\phi$, one begins by assuming ϕ. If one subsequently succeeds in deriving a contradiction $\psi \wedge \neg\psi$ (say), then one is licensed to infer $\neg\phi$.[74] Again, there is a restriction circumscribing the rule: one is allowed to infer $\neg\phi$ from a derivation of $\psi \wedge \neg\,\psi$

[72] Provided we assume standard semantics; but see Fine [44].
[73] On assumptions, see below.
[74] A *contradiction* is any sentence having the same form as $\psi \wedge \neg\psi$.

from ϕ provided that all the assumptions that where introduced after the assumption ϕ was introduced are discharged.

This time, too, we may say that it is an essential part of the inference that one first produces a derivation of $\psi \wedge \neg\psi$ from the assumption ϕ, the axioms cited in the proof, and the assumptions still active at the point $\psi \wedge \neg\psi$ is inferred. Using the same notations as above, we may suggestively write an application of the introduction rule for the negation as follows:



Finally, we have the elimination rule for the existential quantifier. The idea underlying an inference in accordance with this rule is as follows. Given is that $\exists x\phi(x)$. If one succeeds in deriving the sentence $\phi(a) \rightarrow \psi$ for some arbitrary individual constant a, then one is allowed to conclude ψ. The required arbitrariness of a means that the rule is circumscribed by a restriction: the constant a must not occur in an axiom, in an assumption that is still active at this point, in the sentence $\exists x\phi(x)$ and in ψ.

Accordingly, we may again say that ψ is inferred from a specific derivation of $\phi(a) \rightarrow \psi$ from $\exists x\phi(x)$, the axioms, and the still active assumptions. Again writing these axioms as $\alpha_1,\ldots, \alpha_n$ and the assumptions as $\beta_1,\ldots, \beta_n$, we may suggestively write this inference as follows:

$$\boxed{\begin{array}{l}\alpha_1\\ \vdots\\ \alpha_n\\ \vdots\\ \beta_1\\ \vdots\\ \beta_m\\ \vdots\\ \exists x\phi(x)\\ \vdots\\ \phi(a)\rightarrow\psi\end{array}}$$
$$\overline{\psi.}$$

We have now discussed all the introduction and elimination rules of a system of natural deduction for the logical constants $\rightarrow$, $\wedge$, $\vee$, $\neg$, $\forall$, and $\exists$. Let us end up this section by focusing on constructive logical inferences from a procedural point of view.

As an example, reconsider an introduction of the material conditional $\phi \rightarrow \psi$. We have said that this conditional is inferred from a *derivation* of ψ from ϕ, the axioms cited, and the assumptions still active at the point $\phi \rightarrow \psi$ is inferred. However, from a procedural point of view, it would seem not inappropriate to say more specifically that the conditional is inferred from (a run of) a certain (derivational) *procedure* of ψ from ϕ. This gives the inference a strong dynamic flavor. Indeed, from a dynamical point of view, it would seem not inappropriate to say that the conditional is inferred from a run of a derivational *procedure* of ψ from ϕ. Even more so, we may perhaps say that to infer the conditional $\phi \rightarrow \psi$ in some sense *is* to infer ψ from the assumption ϕ. For example, Tomassi [158], p.58, seems to come close to this idea when he explains the introduction of the material conditional as follows: "[…] in general a [material] conditional will have been justified if, having assumed the antecedent, the consequent can also be shown to hold."[75] A similar point holds for the other annotated inferences.

Motivation. Our motivation for distinguishing between simple and annotated inference is twofold.

---

[75] Gamut [53], p.20, seems to play with a similar idea when he suggests interpreting the material conditional as derivability (which, according to a more orthodox conception, should be considered to be a meta-systematic notion).

First, it will turn out that a certain feature of annotated inferences is reflected in the inferences that, in Kant's view, the mathematician carries out in The Passage—or so we shall argue (§ 4.2). The point turns precisely on the fact that, in case of annotated logical inferences, in order for a conclusion to be drawn, it is essential that a derivation be produced first. Though a derivation is built from propositions, it is itself not a proposition. Now, it will turn out that, in Kant's view, in order for a mathematician to draw a conclusion, it is essential for him also to produce (or better: to *construct* in the sense of Kant; § 3.3) a non-propositional item. It is precisely here that annotated inferences have some similarities with mathematical inferences according to Kant.

The point is this: in Kant's view, mathematical reasoning turns out to be mainly a form of non-propositional reasoning (i.e., diagrammatic reasoning). In this respect, a Kantian mathematical proof differs from a logical proof. However, when carefully considered, certain forms of non-propositional reasoning also turn out to be present in logical proofs. (We may refer to this form of non-propositional logical reasoning as "derivational reasoning.")

However, we should hasten to add that the suggested analogy between annotated inferences and mathematical inference according to Kant should not be taken too far. In many other respects, the mathematical inferences according to Kant are radically different from annotated logical inferences—indeed, from logical inferences generally. First, in Kant's case, the item produced is not a derivation but an *intuition*. An intuition is an item of knowledge that is *diagrammatic* in nature—or so we shall see (§ 3.2, § 3.4).[76] In the light of this, "derivational reasoning" is something quite difference from "diagrammatic reasoning." In particular, diagrammatic reasoning employs *spatial* features or characteristics of diagrams. However, a derivation is not in the possession of spatial characteristics (cf. § 2.1). Hence, these cannot be employed in the case of derivational reasoning. Second, we will also see that Kant sees them as distinctly *mathematical* inferences. This gives them an air of "locality" that logical inferences (and hence, annotated logical inferences) lack (cf. § 4.2).

Second, Beth [16], [17], [18] (especially chapter 4), and Hintikka [72], [73] believe that there is no deep conflict between Kant's views on mathematical proof and the logical conception of proof. In their view, a Kantian mathematical proof can be adequately characterized in terms of a logical proof relative to a system of natural deduction. For example, Beth [18], pp. 47-8, holds that the proof described in The Passage accords well a logical proof along the following lines:[77]

[76] See also Lindsay's example inference to be discussed at the end of this section.

[77] Beth [17], pp.19-24, provides a (lengthy) logical proof of the same theorem in full detail. Cf. also Beth [16], pp.372-4.

1.	*Axioms*	
2.	$triangle(ABC)$	assumption
.	.	
.	.	
.	.	
k.	$right\_angles(ABC)$	
$k+1$.	$triangle(ABC) \rightarrow right\_angles(ABC)$	I$\rightarrow$(2, k)
$k+2$.	$\forall x(triangle(x) \rightarrow right\_angles(x))$	I$\forall$($k+1$).

Here, *Axioms* is a sentence abbreviating the conjunction of a suitable collection of axioms for Euclidean geometry. Following Beth,[78] Hintikka [72], [73] also suggests that the proof as described in The Passage can be adequately characterized in terms of a logical proof in a system of natural deduction such as the one above.

The idea is that a mathematician starts with a suitable collection of axioms. Subsequently, he assumes that *ABC* (say) is a triangle (line 2). He then infers, using other assumptions along the way, that the sum of *ABC*'s internal angles is equal to two right angles (line k). Now it can be concluded that *if ABC* is a triangle, then the sum of *ABC*'s internal angles is equal to two right angles (line $k + 1$). Finally, the introduction rule for the universal quantifier is applied, and it is accordingly inferred that everything which is a triangle is such that the sum of its internal angles it equal to right angles (line $k+2$).

Note that Kant opens The Passage with "He [i.e., a mathematician] begins at once to construct a triangle." Beth represents the product of the constructive procedure accordingly carried out as an individual constant (i.e., *ABC*). This individual constant, in turn, figures in the context of a sentence, i.e., an assumption. It will turn out that Hintikka's reconstruction of Kant's notion of intuition finds much inspiration on this point of Beth's (§ 3.2).

Both Beth and Hintikka acknowledged certain constructive elements in Kantian mathematical proofs. Beth has linked these to the introduction rule for the universal quantifier (cf. especially Beth [16], p.365). This is a rule of natural deduction that, in our terminology, is an annotated logical inference (see below). Hintikka has connected the constructive element in Kantian mathematical proofs to those rules of natural deduction that introduce new individuals in the course of the proof (cf. Hintikka [76], p.136). This implies that, in Hintikka's view, the constructive elements in Kantian mathematical proofs should be connected to the

[78] Hintikka [73], p.182, n.7, pays dept to Beth's reading of Kant.

elimination rule for the existential quantifier (again, see below).[79] Both the introduction rule for the universal quantifier considered by Beth and the elimination rule for the existential quantifier considered by Hintikka will turn out to be rules for carrying out annotated inferences.

In contrast with Beth and Hintikka, our considerations suggest that if there is any correspondence between Kantian mathematical inferences and logical inferences at all, this correspondence turns precisely on the annotated logical inferences. The correspondence can be best understood in terms of an analogy: both are forms of non-propositional reasoning. However, it will turn out they are nevertheless quite different types of non-propositional reasoning: we disagree with Beth and Hintikka that a Kantian mathematical proof can on the whole be adequately characterized in terms of a logical proof presented in terms of a system of natural deduction.

§ 2.4. Discovery and justification

With hindsight, we think that the logical conception of proof is to a considerable extent motivated by an important distinction from the philosophy of science, namely, the distinction between context of discovery and context of justification (henceforth often simply referred to as *the distinction*). In the present section, we critically discuss this distinction.[80] The issue is of interest since, since, from a contemporary perspective, Kant may be said to have rather different views on it. In fact, it may be said that Kant to a considerable extent does not accept the distinction between discovery and justification. For him, both go often more or less hand in hand. This is an important point to realize, since it will influence our appreciation of many things Kant has to say to us.

We begin this section by introducing the distinction in relation to logic. Next, we argue that the distinction is often understood in different though related ways, thereby basing ourselves on some useful work done by the philosopher of science Paul Hoyingen-Heune [82]. We subsequently argue that context of discovery and context of justification are often much closer than proponents of the distinction sometimes tend to suggest. Finally, we point out that it is not always clear *what*, precisely, it is that is supposed to be "discovered" within what is considered the context of discovery. In this respect, we make some distinctions and offer a few examples.

---

[79] It turns out that Hintikka specifically concentrates attention on the elimination rule for the existential quantifier. The elimination rule for the existential quantifier does not seem to play an explicit role in Hintikka's considerations.

[80] Since a couple of decades, the distinction has become to stand under attack by philosophers of science. See, for example, Nickles [117].

Logic and the context distinction. The distinction between context of discovery and context of justification can be considered as one of the more influential contributions of 20th century logical empiricism. In this respect, the distinction is often attributed to Reichenbach [134], chapter 1, § 1. Earlier sources might be mentioned, however. See for example Frege [47], p.5, Frege [45], pp.3-4. Furthermore, Popper [126], p.31, wonders whether the distinction can already be found in Kant under the guise of the distinction between *quid iuris* and *quid facti* (cf. A84/B116).[81] See also Hoyningen-Heune [82], pp.502-3, for further historical references.

Even up to these days, many logicians appear to accept the distinction between discovery and justification almost as an article of faith. Thus, without spending many words on the issue, we read the following in a recent edition of a fairly well-known textbook:

> In general, logic does not deal with the context of discovery. The mental processes used in thinking of hypotheses or conclusions are of interest to the psychologist, not the logician. The logician is interested in reasons that are, or might be, presented in support of conclusions. In other words, the logician is interested in the context of justification (Tidman and Kahane [157], p.15).

(See also Copi [31], pp.4-5; Salmon [141], pp.10-1.)

It appears that Tidman and Kahane endorse a very broad conception of logic, incorporating propositional logic, predicate logic, epistemic logic, probability theory and set theory, among other things. Besides that, their book also contains chapters on, for example, the theory of definitions (chapter 10), the theory of explanation and the theory of confirmation (chapter 16), the analytic and the synthetic (chapter 19), and axiomatics (chapter 20).[82]

In the light of this, the authors manifest a strong tendency to conflate logic and scientific methodology (cf. Tidman and Kahane [157], p.249). This may very well be considered a characteristic trait of the tradition stemming from the logical empiricists. In particular, within this tradition, the methodology of mathematical proof is seen as the method of logical proof. As to the logical empiricist tradition, views along these lines where defended by, for example, Carnap [26], Hempel [66], and Hahn [61]. But surely Frege [46], too, can be taken as an earlier precursor of this view. The quote given above shows that this view still has currency today. See also § 2.2.

Hoyningen-Heune's analysis of the distinction. The philosopher of science Paul Hoyningen-Heune [82] has pointed out that the distinction between context of discovery and context of justification is often understood as a complex of

[81] We shall not attempt to answer Popper's worry here.
[82] We simply refer to Tidman and Kahane [157] where the reader can find all the issues addressed.

different though related distinctions. Sorting these out has lead Hoyningen-Heune to a list of five (see below).[83]

Hoyningen-Heune suggests that the distinction does not simply fall apart into five different distinctions. In contrast, what he means to say is that various proponents of the distinction have often implicitly understood it as a combination of various related distinctions. And not just by mistake, Hoyningen-Heune says, but much more as a consequence of certain philosophical presuppositions (cf. *ibid*., p.504).

We shall present Hoyningen-Heune's list below and elucidate the respective items that occur in it. Not all of them are equally relevant. Nevertheless, for the sake of completeness we present them all. A critical discussion follows subsequently.

1. ***Two types of processes*****.** The distinction between discovery and justification might be understood as the distinction between two types of processes. For the purposes of the present work, relatively short-time psychological processes are especially relevant.

Processes of discovery and processes of justification are typically understood to be temporally disjoint processes. More precisely, the process of justification is often supposed to take place *after* the process of discovery (see our discussion below for examples).

2. ***Processes of discovery vs. methods of justification*****.** In this second sense, the distinction between discovery and justification is understood as the distinction between the process of discovery and the methods of justification or the reconstruction of that which confers justification. According to Hoyningen-Heune, in the present sense the distinction between discovery and justification is understood as one between the factual and the normative respectively (*ibid*., p.504).

Hoyningen-Heune furthermore points out that, when formulated in this manner, no distinction is made between methods or reconstructions of justification that have been endorsed at some time in the past on the one hand and present or "eternal" standards on the other (*ibid*.). This point is dealt with next.

3. ***Processes of discovery vs. logic*****.** Understood in this way, the methods of justification or the reconstruction of justification is considered the business of logic, for example, logic in the modern, post-Fregean era.[84] Other methods or ways of reconstruction that have been endorsed in the past do not deliver justification. What counts as a justification is an eternal affair and is accordingly time-independent.

---

[83] In fact, the distinction has been understood in different ways, or in terms of a combination of different ways (see *ibid*., p.504).
[84] Hoyningen-Heune evidently endorses a broad conception of logic.

4. *Logic vs. psychology* (*and sociology and history*). Understood in this way, the distinction concerns one between several academic disciplines. On the one hand, it would be logic addressing the methods or reconstruction of justification. On the other hand, a discipline such as psychology (or sociology, or history) would be concerned with matters that one could put under the heading of discovery.

The relation between logic on the one hand and psychology on the other is not symmetric. Thus, psychology can learn from logic what the method of justification of, for example, a discovery belonging to the field of psychology is. In the other direction, there is nothing to learn for a logician from the findings of a psychologist (or a sociologist, or a historian). The correctness or adequacy of a method or reconstruction of justification is a matter of logic alone.

5. *Types of questions*. Hoyningen-Heune points out that the distinction between discovery and justification is often introduced by means of distinguishing between the types of questions that might be asked about discovery and the types of questions that might be asked about justification (see Hoyningen-Heune [82], pp.505-6, including the references given there). One the one hand, for example, it may be asked how it is that one comes to a certain discovery. On the other hand, it may be asked whether the discovery can be justified, and if so, how. Subsequently, one frequently turns to distinctions that seem to be "implied" (as Hoyningen-Heune puts it) by the distinction between these two types of questions. One distinction that may be taken as being implied by these two types of questions is the distinction between context of discovery and context of justification.

Discussion. Items 1-4 are particularly relevant for us. They are respectively discussed below.[85]

1. The distinction between discovery and justification understood as a distinction between two types of processes can be illustrated by way of cases that belong to the folklore of the history of science (cf. Salmon [141], p.11).

The first example concerns the famous Indian mathematician Srinivasa Ramanujan. Thus, the editors of Ramanujan's collected papers report that the latter was often prompted mathematical theorems by the Goddess of Namakkal during his sleep (Ramanujan [131], p.xii). Presumably, Ramanujan's sudden hunches do not count as proofs. Proving the theorem, we may suspect, is something that takes place after the divine communication.

The second example concerns Henri Poincaré, another and no less famous mathematician. In a lecture to the Société de Psychologie in Paris, Poincaré reports a well-known case concerning discovery in mathematics (Poincaré [123],

[85] For a discussion concerning item 5, see Hoyningen-Heune [82], pp.511-2.

p.387). Poincaré tells us that he once took part in a geological conference. During the conference, Poincaré recalls, his mind was occupied with matters completely different from those on his day-to-day mathematical work. On one day during the conference, a group of participants, including Poincaré himself, went for a bus-drive. While stepping on the footboard of the bus that would bring the group to the place of destination, Poincaré suddenly and seemingly out of the blue saw the light on a mathematical problem he had been pondering over for days before the conference. He ends his story by telling us that he proved the theorems that his discovery led to a few days after this remarkable event.[86]

Wesley Salmon has pointed out, however, that one should realize that such cases of discovery in terms of dreams or proverbial "flashes of light," though certainly interesting, are nevertheless often somewhat dramatic and perhaps a-typical to some extent (Salmon [140]). In any case, these examples do not entitle us to conclude that processes of discovery and processes of justification are in general temporally disjoint. There are situations where the process that lead one to a discovery virtually goes hand in hand with the justification of what is discovered accordingly.

In order to illustrate this, Salmon draws attention to those cases where one comes to know something by executing what he calls a "sound algorithmic procedure" (cf. *ibid.*). Though Salmon does not give any concrete examples, we can easily come up with one by ourselves. Thus, consider the simple case where one is requested to find (i.e., to come to know) the greatest common divisor of two integers. To this end one may apply, for example, Euclid's algorithm. One's knowledge that the greatest common divisor of the integers given is actually the number found by the application Euclid's algorithm, is intimately related to certain characteristics of this algorithm. Among these characteristics is that applying the algorithm always does what it is designed for: it delivers the greatest common divisor of any pair of integers that is given as input. In a case like this, it seems hard to tell apart the process of discovery and the process of justification. In a sense, the process of discovery appears virtually identical with process of justification.

2. In the second sense, the discovery/justification dichotomy was understood as a distinction between processes of discovery on the one hand and the methods or reconstruction of justification on the other.

As opposed to this, one might wonder, however, whether and to what extent there would be certain methodological aspects connected processes of discovery too. Within the philosophy of science this has led to an investigation into "logical aspects of discovery" (in a broad sense of *logical*). While this idea is by no means considered unproblematic (cf. Hoyningen-Heune [82], p.507, including

[86] It should be noted that within the context of the present work, this example is strictly speaking somewhat beside the mark. For considering Poincaré's own words, the "discovery" he in fact made on this occasion was not a theorem or a proof but a definition. (This was pointed out by Jean-Paul van Bendegem.)

the references given there), it would nevertheless seem plausible that there are several systematic aspects underlying at least a certain restricted class of processes of discovery.

Related to this, solving mathematical problems evidently involves elements of discovery. In the light of this, the mathematician George Polya [124], [125] has put such systematic methods to the end of solving mathematical problems under the general heading of "plausible reasoning." For example, Polya points out that reasoning by way of analogy or induction (i.e., making certain generalizations on behalf of a restricted class of specific cases) may be promising when one tries to solve problems in mathematics.

However, not only mathematical problem solving involves elements of discovery. A similar point holds for problem solving generally. In this respect, we should mention the work on problem solving that has been undertaken in cognitive science and AI. See, for example, Newell and Simon [116] for groundbreaking work in this area.

3. In the third sense, the distinction between discovery and justification was understood as the distinction between processes of discovery and logical methods of justification or reconstruction of justification. The difference is that the methods of justification, or reconstruction of justification, are now considered eternal instead of time-dependent. Nevertheless, the same points can be made here, with the only difference that they now apply to eternal standards of justification.

4. Finally, in the fourth sense, the distinction between discovery and justification was understood in terms of a distinction between various academic disciplines. In particular, some proponents of the distinction are inclined to think that the relation between logic and psychology is asymmetric. The findings of a psychologist are not relevant for the logician but not necessarily *vice versa*.

This way of understanding becomes especially relevant when we turn our attention to the anti-psychologism, a theme that seems so dominant within the tradition of modern logic.

Originally, Frege and other anti-psychologists claimed that the method of justification is not a matter of psychology (cf. Frege [45], especially the preface). Now while Frege certainly seems right here, we must not readily conclude that psychological considerations simply are of no relevance for the method of justification or the reconstruction thereof. For example, given a certain method of proof, considerations of complexity may lead one to conclude that it is practically impossible to prove a theorem. Therefore, the conclusion at this point is that either justification is impossible to obtain in practice, or that justification is less intimately related to proof than is initially suggested. Accordingly, it is not clear that considerations bearing on an agent's available cognitive resources are irrelevant for what the method of justification is.[87]

Discovery of what? One question we have as yet not considered is *what*, exactly, it is that one discovers when one discovers something.

Some seem to have thought that discovery in general concerns the discovery of a proposition.[88] However, proofs, too, are things that might be discovered. For example, how logical proofs might be discovered in a way that can be emulated by a computing machine forms an important concern of a branch of AI called *automated theorem proving* (see Gelernter [54] for an early contribution to this field; see also Jamnik [85], chapter 1).

In the light of this, the following point is especially relevant for the purposes of the present study. Considering, for example, The Passage, we notice that even a proof itself might involve certain elements that might be put under the heading of discovery. Thus, Kant describes a proof as a procedure that involves, among other things, the addition of several auxiliary lines to a triangle. The production of these auxiliaries and putting them in the right place is of crucial importance for the proof. One might reasonably suggest that the production of these auxiliaries has certain aspects that pertain to the context discovery. Furthermore, one might even propose that the production of the triangle in the earliest stage of the proof is an act of discovery (or, as the case may be, invention) to begin with.

The upshot of this brief discussion is that if we look at proofs from a procedural point of view, then the boundary between discovery and justification starts to blur. A proof may itself involve various elements of discovery.

§ 2.5. Conclusion

In this chapter, we have investigated the logical conception of proof. According to the logical conception of proof, a mathematical proof is (ideally) a logical proof relative to a system of natural deduction. By way of conclusion, let us state the following four points. First, according to the logical conception of proof, proving a theorem fundamentally takes place by way of propositions. One proves a theorem by inferring propositions from other propositions by applying logical rules of inference to them. This point is intimately related to the fact that the logical conception of proof sees language (sentences) as the prime medium of reasoning. Propositions are expressed only by sentences. Second, an important philosophical idea lying at the background of the logical conception of proof is

[87] See Rood [135] for a discussion of psychology-related factors that might be relevant for matters of justification.

[88] For example, Hoyningen-Heune clearly suggests some such thing when he without further ado speaks in terms of someone having "discovered that *p*"; Hoyningen-Heune [82], p.508.

the close relationship between logic and scientific methodology. In particular, the method of mathematical proof is typically taken to be the method of natural deduction. Third, and related to this, we argued that the logical conception of proof is to a considerable extent motivated by the distinction between context of discovery and context of justification. However, we have argued that elements of discovery and justification are often subtly and delicately interrelated. Thus, the logical conception of proof loses a considerable part of its initial motivation. Fourth, upon closer inspection of the rules of natural deduction, we where able to distinguish between two types of logical inference: simple logical inference and annotated logical inference. In contrast with a simple logical inference, an annotated logical inference involves a constructive element of a sort: in order to draw a conclusion, one first has to produce (or construct) a certain derivation. We have suggested that if it makes sense at all to seek for certain similarities between Kant's views on mathematical proof and the logical conception of proof (and any such similarity should be taken with care), it is precisely on this point—that construction plays a role in proving a theorem. All the themes raised will return in later chapters.

Chapter 3

Mathematical ways: Kantian construction and intuition

In the present chapter, we discuss the most central and important elements of Kant's philosophy of mathematics, namely, the notion of construction and the related notion of intuition. As it turns out, both these elements play a key role in Kant's philosophical underpinning of the method of proof exemplified in The Passage (see chapter 4).

The outline of this chapter is as follows. We begin by providing some background by briefly considering Locke's views on proof (§ 3.1). Next, in § 3.2, we turn to a discussion of Kant's notion of intuition. In § 3.3, we consider the related notion of construction. In § 3.4, we return to Kant's notion of intuition and bring out the diagrammatic structure of intuitions. We end up with stating our conclusions (§ 3.5).

§ 3.1. Background: Locke on reasoning as reasoning with ideas

In the present section, we discuss Locke's views on proof. Our aim is to argue (1) that for Locke, reasoning is primarily reasoning with ideas, and (2) that, according to Locke, logic is not a useful instrument for proving theorems. The first point may be taken to challenge the idea that reasoning is propositional. The second raises questions concerning the relation between logic and the methodology of mathematical proofs.

Locke's views on proof are of interest since they exemplify an interesting stepping-stone towards Kant. Locke's point that a mathematician reasons with ideas can be considered as something to which Kant adds interesting and important new elements. This will become clear in § 3.2.

We begin with considering how, in Locke's view, a mathematician proves theorem 1 from § 1.1. Subsequently, we discuss Locke's suggestion that this proof is to be understood primarily as a succession of ideas. Next, we argue that such a succession of ideas can nonetheless be *reconstructed* as a train of syllogisms. Finally, we discuss Locke's views on the relation between proof and logic.

Before we start, we would like to emphasize that we are only interested in certain general traits of Locke's views on proof. We realize that there are many details to be desired in what follows. It is not even clear whether Locke's views can be maintained when put under closer scrutiny. However, this is not an issue deserving our main interest. Our point is to highlight that certain developments in logic are not merely developments towards more richer and expressive logical systems (e.g., first-order logic as an improvement of traditional syllogistic logic). In many respects, these more formal developments mark fundamental shifts in our thinking of what we take a proof to be. Related to this, these developments were also accompanied by a change in view concerning the role and place of logic in the process of reasoning and proof.

One of the functions of the current section is to lead our attention away from a conception of proof that exclusively sees the proposition as the fundamental unit of cognitive organization. Furthermore, this section forms an interesting stepping-stone towards our discussion of Kant's view of proof (cf. the rest of this thesis). It is illuminating to think of Kant's views on proof as constituting a further development of a view as exemplified by Locke.

Proof according to Locke. Recall that in The Passage Kant describes how *he* thinks a mathematician proves that the sum of the internal angles of a triangle equals two right angles (§ 1.1, theorem 1). Strikingly, in his *Essay concerning human understanding* (Locke [107]), Locke also gives a description of the way a mathematician proves this theorem (clarification will be given shortly):

> […] the Mind being willing to know the Agreement or disagreement in bigness, between the three Angles of a Triangle cannot by an immediate view and comparing them, do it: Because the three Angles of a Triangle cannot be brought at once, and be compared with any other one, or two Angles; […] In this Case the Mind is fain to find out some other Angles, to which the three Angles of a Triangle have an Equality; and finding those equal to two right ones, comes to know their Equality to two right ones.
>
> Those intervening ideas, which serve to shew the Agreement of any two others, are called Proofs; and where the Agreement or Disagreement is by this means clearly perceived, it is called Demonstration, it being shewn to the Understanding, and the Mind made see that it is so (Locke [107], p.532).

Locke's formulations are somewhat odd and confusing. Let us provide some clarification.

The gist of the quotation above is reasonably clear. In the above quotation, Locke considers an agent ("the Mind") who is willing to know "the Agreement or disagreement in bigness, between the three Angles of a Triangle." In other words, what Locke considers is an agent who desires to know what the sum of the internal angles of a triangle is. This becomes obvious when we consider the

61

rest of the quotation. What the agent considered by Locke eventually comes to know (or proves), of course, is that the sum of the internal angles of a triangle is equal to two right angles. This is theorem 1 from § 1.1.

Interestingly, for Locke, to prove this theorem means to show that a certain connection ("agreement") obtains between two ideas. More specifically, we may say that to prove this theorem means to show that the idea of the (sum of the) three angles of a triangle and the idea of (the sum of) two right angles in are in some sense connected to one another. These exact wordings may not be precisely clear from Locke's own formulations. Nevertheless, we think this come down to a reasonable interpretation of what he says.

As did Kant in The Passage, Locke, too, describes his proof first of all as a certain cognitive procedure. Like Kant, Locke is considering an agent whose goal is to prove (and hence to come to know) a theorem. Accordingly, Locke orientation towards proof is primarily cognitive in nature. It turns out, however, that Locke's views on the structural organization of a proof process are somewhat different from Kant's. A full appreciation of this point can only be achieved at the end of chapter 4. In the meantime, let us first consider Locke's view on the issue somewhat more closely.

Reasoning with ideas. In the quotation cited above, Locke says that an agent cannot come to know immediately that the sum of the internal angles of a triangle is equal to two right angles. In other words, an agent cannot immediately see that a connection obtains between the two ideas mentioned. What Locke subsequently suggests is that in order to show this connection, a mathematician attempts to find several intervening ideas. "Those intervening ideas," Locke says, "are called Proofs." Furthermore, when those intervening ideas make the connection between two ideas "clearly perceived," Locke together calls them a *demonstration*.

What, in Locke's view, is precisely an idea? We will not pursue this question in any detail, since this would quickly deter our attention away from the main purpose of this study (see, e.g, Chappell [29]).[89] Instead, let us simply quote Locke:

> […] the word *Idea* […], I think, serves best to understand whatsoever is the Object of the Understanding when a man thinks (Locke [107], p.47).

An idea is whatever is the object of the understanding when an agent thinks. Connecting with the terminology used in the present work, we may say that, in Locke's view, an idea is an item of knowledge (broadly understood). In order to give some examples, Locke says that ideas

---

[89] Locke devotes the entire Book II of his *Essay concerning human understanding* to a discussion of ideas (see *ibid*.).

> […] are those expressed by the words, *Whiteness*, *Hardness*, *Sweetness*, *Thinking*, *Motion*, *Man*, *Elephant*, *Army*, *Drunkenness*, and others […] (*ibid*., p.104).

We henceforth write ideas in `courier`. For example, the idea of a triangle is written as `triangle`, the idea of a circle is written as `circle`.

In order to facilitate the discussion of the present section, let `A` abbreviate the idea of a triangle (or the idea `triangle`). Let `B` abbreviate the idea of being such that the sum of the (three) internal angles is equal to two right angles (i.e., the idea `the sum of the (three) internal angles being equal to two right angles`).

Locke's point is that, in order to prove that the sum of the internal angles of a triangle is equal to two right angles, a mathematician seeks for several ideas intervening between `A` and `B`. Let us refer to these ideas as $\mathtt{C}_1,\ldots,\mathtt{C}_n$. We suspect that these ideas can be best understood as forming a *succession*. Accordingly, the way they intervene between `A` and `B` can be depicted as follows:

$$\mathtt{A} \longrightarrow \mathtt{C}_1 \longrightarrow \mathtt{C}_2 \longrightarrow \quad \ldots\ldots \quad \longrightarrow \mathtt{C}_n \longrightarrow \mathtt{B}.$$

The impression we get is that, in order to prove the aforementioned theorem (and similar theorems), one starts with the idea `A`, makes several successive transitions to intermediate ideas $\mathtt{C}_i$, until one arrives at the idea `B`.[90]

For convenience, we refer to a succession of ideas $\mathtt{C}_1,\ldots,\mathtt{C}_n$ aiming to establish a connection between ideas `A` and `B`, a *Lockean proof with respect to* the ideas `A` and `B`, or a *Lockean proof* for short.

That said, we can now make the following point: compared with a modern logic-oriented conception of proof (see chapter 2), Locke has different views as to what the fundamental building blocks of a proof are. While modern logic thinks that the items of knowledge an agent reasons with are at the fundamental level propositional, Locke thinks that they are formatted in terms of ideas. What we have before us, then, is a different view on the internal organization of a proof. Let us address a final issue that will be of relevance for the next section.

Besides ideas, Locke also acknowledges propositional knowledge, which he explains as "the perception of the certain agreement, or Disagreement of two *Ideas* (cf. Locke [107], p.685).[91] In other words, propositional knowledge is a

[90] See Descartes [36] (especially p.25), for a strikingly similar view.
[91] To be more precise, Locke distinguishes between two types of propositional knowledge: *intuitive knowledge* and *rational knowledge*.

63

perception of a relation—a relation of agreement or disagreement—between two ideas. For Locke, then, propositional knowledge in some way *involves* ideas. That is why for Locke ideas form the principal type of item of knowledge. With respect to ideas, the proposition may be considered an item of knowledge in a derived sense.

Locke himself no doubt believed that typical propositions are categorical propositions. Indeed, this was quite commonly accepted among philosophers in Locke's time. Accordingly, it seems reasonable to conclude that the two ideas involved in a proposition are precisely the subject idea and the predicate idea of a categorical proposition. In the light of this, a proposition is affirmative when the two ideas "agree"; a proposition is negative if the two ideas "disagree." In order to simplify the discussion, we henceforth only consider affirmative propositions. When two ideas are "made to agree," then we think of this as if an operation is applied to them. We write this operation as *is* (in italics). A (affirmative) proposition with subject idea S and predicate idea P is written as S *is* P.[92]

Accordingly, given the notations introduced earlier, the proposition that the sum of the internal angles of a triangle is equal to two right angles can be written as A *is* B. The Lockean proof $C_1, \ldots, C_n$ described above is a proof of the theorem A *is* B. What we see, then, is that in general, a Lockean proof connects the subject idea and the predicate idea of a categorical proposition. Perhaps we may say that a Lockean proof *shows* these two ideas to agree.

Syllogistic reconstruction of Lockean proofs. A Lockean proof can be easily reconstructed as a series of syllogisms. Let us see how such a reconstruction might go.

Recall that A stood as an abbreviation for the idea of a triangle; B stood as an abbreviation for the idea of the sum of the (three) internal angles being equal to two right angles. Conceived of as a categorical proposition, both the ideas A and B are respectively associated with the subject term and the predicate term of theorem 1 from § 1.1. For convenience, let us consider theorem 1 again:

> *Intuitive knowledge*, is the perception of the certain Agreement, or Disagreement of two *Ideas* immediately compared together. *Rational knowledge*, is the perception of the certain Agreement, or Disagreement of any two *Ideas*, by the intervention of one or more other *Ideas* (Locke [107], p.685).

(Besides that, Locke also acknowledges *judgment*, which he thinks of as a kind of probable belief; cf. *ibid*.) Note that, in Locke's view, rational propositional knowledge and proof are two very closely related notions. Insofar as our purposes are concerned, Locke's notion of intuitive knowledge is different from Kant's notion of intuition. From the point of view adopted in the present study, they differ in that Locke's intuitive knowledge is primarily a form of propositional knowledge, while Kant's intuitions are primarily non-propositional items of knowledge. See the rest of this chapter for more details on Kant's notion of intuition.

[92] In order to make the discussion not unnecessarily complicated, we ignore the quantity of categorical propositions.

The sum of the internal angles of every triangle equals two right angles.

In order to make the aforementioned categorical structure of this proposition more vivid, let us rewrite it as follows:

Every triangle is such that the sum of its internal angles is equal to two right angles.

Conceived of as an item of propositional knowledge, let us write this theorem as $\mathtt{A}$ *is* $\mathtt{B}$.

A train of ideas as depicted in the diagram above can be reconstructed as a series of syllogisms, as follows. When the mathematician has made a transition from the idea $\mathtt{A}$ to the idea $\mathtt{C}_1$, we may say that he has proved the proposition $\mathtt{A}$ *is* $\mathtt{C}_1$. When the mathematician has made a transition from $\mathtt{C}_1$ to $\mathtt{C}_2$, we may say that he has established the proposition $\mathtt{C}_1$ *is* $\mathtt{C}_2$. The proposition $\mathtt{A}$ *is* $\mathtt{C}_2$ can now be inferred.

Taken together, we can say that the idea $\mathtt{C}_2$ corresponds to the "middle term" of the following syllogism:

$$\frac{\begin{array}{c}\mathtt{A}\ \textit{is}\ \mathtt{C}_1\\ \mathtt{C}_1\ \textit{is}\ \mathtt{C}_2\end{array}}{\mathtt{A}\ \textit{is}\ \mathtt{C}_2}$$

(cf. also Locke [107], pp.673-5).

Collectively, we can reconstruct the following series of syllogisms from the original proof (i.e., the original succession of ideas $\mathtt{C}_1,\ldots,\mathtt{C}_n$):

$$\left.\begin{array}{r}\mathtt{A}\ \textit{is}\ \mathtt{C}_1\\ \mathtt{C}_1\ \textit{is}\ \mathtt{C}_2\end{array}\right\}\rightarrow \left.\begin{array}{r}\mathtt{A}\ \textit{is}\ \mathtt{C}_2\\ \mathtt{C}_2\ \textit{is}\ \mathtt{C}_3\end{array}\right\}\rightarrow\ \mathtt{A}\ \textit{is}\ \mathtt{C}_3$$

$$\ldots$$

$$\ldots\quad \rightarrow \left.\begin{array}{r}\mathtt{A}\ \textit{is}\ \mathtt{C}_n\\ \mathtt{C}_n\ \textit{is}\ \mathtt{B}\end{array}\right\}\rightarrow\ \mathtt{A}\ \textit{is}\ \mathtt{B}.$$

We call the above series of syllogisms the series of syllogisms *associated* with the (original) Lockean proof $\mathtt{C}_1,\ldots,\mathtt{C}_n$ (of the proposition $\mathtt{A}$ *is* $\mathtt{B}$). What we have shown is how, in general, an associated series of syllogisms can be reconstructed from any Lockean proof.

Given the close connection between a Lockean proof and an associated series of syllogisms, it would seem that for Locke, reasoning could *also* be seen as reasoning with propositions.[93] However, reasoning is reasoning with propositions only in a derivative sense: one first needs to *reconstruct* an original proof as a series of syllogisms. This is why we said that for Locke, reasoning is *primarily* reasoning with ideas.

Note that important logical notions such as validity and soundness (cf. § 2.1) can be straightforwardly applied to the above series of syllogisms. In fact, both notions apply without problems to any inferential step that can be discerned in this series of syllogisms.[94] However, the point does not hold in case of the original Lockean proof this series of syllogisms aims to reconstruct. As regards this Lockean proof, both validity and soundness cannot be naturally discerned, and distinguished from one another. The main cause of this is that the items of knowledge reasoned with are ideas. Truth, a central notion figuring in the respective definitions of validity and soundness, does not seem to apply naturally to ideas. Thus, it would seem that a Lockean proof is not simply suitable for logical study. In particular, in order to apply logical notions such as validity and soundness, the original Lockean proof needs to be logically reconstructed first. But perhaps a qualification is in order here.

In case of the above series of syllogisms, one may say that the theorem $\mathtt{A}$ *is* $\mathtt{B}$ is inferred by means of a series of *valid* inferential steps. On top of that, the truth of this proposition will be secured by the *soundness* of the inferential steps. However, in case of the original Lockean proof, every step undertaken is a transition from one idea to another. Starting with the idea $\mathtt{A}$, a transition from one idea (possibly $\mathtt{A}$ itself) to a subsequent idea $\mathtt{P}$ (say) evidently secures the truth of the proposition $\mathtt{A}$ *is* $\mathtt{P}$ ($\mathtt{P} \in \{\mathtt{C}_1,\ldots, \mathtt{C}_n, \mathtt{B}\}$). Accordingly, due to the specific structure of the procedure, such a transition can be seen as being in some sense an inferential step that is valid and sound at once. Therefore, insofar as validity and soundness do apply to Lockean proofs, they go firmly hand in hand.

Lockean proofs and Syllogistic logic. For Locke, an important part of proving a proposition $\mathtt{A}$ *is* $\mathtt{B}$ consists in *finding* intermediate ideas $\mathtt{C}_1,\ldots, \mathtt{C}_n$ that establish the relevant connection between the ideas $\mathtt{A}$ and $\mathtt{B}$. Note that this means that, in Locke's view, a proof is permeated by elements of discovery (cf. Locke [107], p.532). Thus, Locke's view on proof brings with it almost a complete a blurring of the boundary between context of discovery and context of justification (§ 2.3).

Locke holds that to the end of proving a theorem, a reconstruction in terms of syllogistic logic is of no use (cf. Locke [107], pp.672-3.). To find intermediate ideas $\mathtt{C}_1,\ldots, \mathtt{C}_n$ means to find the respective middle terms $\mathtt{C}_1,\ldots, \mathtt{C}_n$ of the series

[93] The assumption is that syllogistic logic is indeed a logic of propositions. This assumption may gain some plausibility if one is inclined to think as syllogistic logic in terms of a fragment of monadic first-order logic. But see also footnote 62.
[94] Strictly speaking, the notions of validity and soundness apply to arguments. However, in the present case, it is obvious how they also apply to inferential steps.

of syllogisms associated with the original proof. However, the very possibility of a syllogistic reconstruction presupposes that one already has those middle terms.

Locke typifies logic in rather negative terms. More specifically, for Locke a series of syllogisms associated to a Lockean proof in terms of “perplexed Repetitions” and a “Jumble” of syllogisms. Furthermore, Locke suggests that the original Lockean proof reveals much more clearly the connection between two relevant ideas than when this proof is “transposed and repeated, and spun out to a greater length in artificial Forms” (Locke [107], p.673).

As a result, Locke holds that syllogistic logic is not the “proper instrument of reason” (cf. *ibid*., p.670). In the light of the foregoing considerations, we think that this might be interpreted as follows: syllogistic logic does not deliver the means for finding the intervening ideas that are so crucial for proving a theorem. Thus, when it comes to proof, logic does not seem to play a very significant role. Very likely, this is intimately related to the fact that Locke was mainly interested in those aspects of proofs that concern their discovery.

We shall not enter into the question what, according to Locke, the proper instrument for proving theorems is. We suffice to note that Locke seems to associate it with a certain mental power (as we may call it) he refers to as “sagacity”:

> A quickness in the Mind to find out these intermediate *Ideas* [i.e., proofs], (that shall discover the Agreement or Disagreement of any other,) and to apply them right, is, I suppose, that which is called *Sagacity* (Locke [107], p.532).

Concluding remarks. In the present section, we have made the following two main points. First, in Locke’s view, reasoning is *primarily* reasoning with certain non-propositional items of knowledge, viz. ideas. This conclusion is not harmed by the fact that a Lockean proof can be reconstructed as a series of syllogisms. For the latter shows that, in Locke’s view, reasoning is propositional reasoning only in a derivative or secondary sense. Second, Locke does not think that logic is instrumental for proving theorems. This raises questions concerning the relations between logic and the methodology of proofs.

Though the points raised in the present section may be interesting and relevant, we are left with the impression that Locke’s views on proof are intimately related to traditional syllogistic logic in many respects. This is particularly because a Lockean proof is so easily reconstructed as a series of syllogisms. In the light of this, it seems hard to see, for example, how Locke would account of the fact that a proof typically involves reasoning with relations (e.g., a relation such as IS PARALLEL WITH). This is no doubt because from the standpoint of modern first-order logic, syllogistic logic is a system of monadic first-order logic. This system does not allow for reasoning with relations.

However, we believe that this does not form a good reason to dismiss Locke's views on proof in their entirety. The points raised in this section stand to some extent independent of matters concerning the expressiveness of syllogistic logic with respect to systems of modern logic. First, to say that reasoning is primarily propositional reasoning is something that now turns out less obvious than it may have seemed to those trained in modern logic. An argument on this score cannot be provided simply by pointing to logical systems in the modern tradition and the advantages they have in terms of expressiveness. For these systems *presuppose* that reasoning is propositional reasoning. Second, Locke opens up questions pertaining to the relation between logic and the methodology of proofs. In particular, though modern systems of natural deduction where surely unknown to Locke, we cannot resist wondering what reasons we have for believing that the way of mathematical proof is indeed the way of natural deduction. The issue at stake is what one takes logic to be. We return to this point in § 5.3.

§ 3.2. Kantian intuitions

Philip Kitcher once said that "'[i]ntuition' is one of the most overworked terms in the philosophy of mathematics" (Kitcher [97], p.49). This puts heavy demands on our attempts to clarify Kant's notion of intuition. The present section aims to fulfill these demands partly. We shall be especially concerned with the more epistemological dimensions of the notion.

This section proceeds as follows. We begin by considering the various items of knowledge Kant has distinguished. Next, we briefly review readings of Kant's notions of intuition that have been defended in the literature (we focus mainly on the views put forward by Jaakko Hintikka and Charles Parsons). Finally, we turn our attention mainly to intuitions and discuss Kant's distinction between a priori and a posteriori intuitions.

Items of knowledge, according to Kant. What Locke calls an idea can be put more or less in the same box as what Kant calls an *Erkenntnis*. In the English translation of the *Critique of pure reason* we are using as our source, Kant's term *Erkenntnis* is translated as *cognition*. However, in line with our own terminology, we use *item of knowledge* instead.

We may say that Locke acknowledged only one fundamental type of item of knowledge: the idea (§ 2.1). Kant, in contrast, held that an item of knowledge "is either an *intuition* or a *concept* (*intuitus vel conceptus*)" (A320/B376; see also Kant [93], p.589).[95] It will turn out that intuitions and concepts can in some sense

[95] Kant did also recognize the judgment as an item of knowledge, but only in a derivative sense; see below.

be considered as the fundamental or prime types of items of knowledge: for Kant, other types of items of knowledge (propositions) or in some way built from concepts or intuitions.

Kant considers an item of knowledge as a kind of representation.[96] More specifically, Kant says, an item of knowledge is a "representation with consciousness" (cf. A320/B377).[97] It may seem plausible to hold that what Kant has in mind here is that an item of knowledge is a representation whose object an agent is conscious *of*.[98] However, there are considerations that certainly do not exclude that the representation itself is also something an agent is conscious of. Indeed, as we will argue shortly, Kant does not seem to make a clear distinction between a representation on the one hand and an object on the other.

Kant does not think an item of knowledge to be a mere representation with consciousness. Something along these lines is already indicated by what we said in the previous paragraph. Explicitly, Kant holds that an item of knowledge is an *objective* representation with consciousness (*ibid*.). We think that, according to Kant, objectivity means just that: having a reference to an object (broadly understood) (cf. A155/B194, A320/B376).[99]

It may be considered remarkable that Kant sometimes tends to speak about intuitions not as representations (of objects) but as objects instead. For example, at some point Kant says that an intuition, "as intuition, is an *individual* object" (A713/B741; the emphasis is Kant's).[100] Perhaps this can be in part explained by pointing out that Kant has a tendency to conflate the object of an item of knowledge refers to on the one hand with the content of that item of knowledge on the other (A55/B79; A58-9/B83; A63/B87; cf. A239/B298). It seems as if Kant thinks that, at least in case of intuitions, content and object do not exist on two separate levels, but are more or less melted together so to speak. Thus, we can begin to understand how it is that an item of knowledge, conceived of as a representation *cum* content, is easily taken for an object.[101]

Seemingly conflicting an earlier point, Kant sometimes also appears to recognize the proposition as an item of knowledge.[102] For example, when he says that "[j]udgment is […] the mediate cognition of an object" (A68/B93). Although Kant's theory of propositions (i.e., judgments) is not particularly relevant for our

---

[96] Kant's own German term is *Vorstellung*.
[97] "Vorstellung mit Bewußtsein."
[98] Therefore, in Kant's view, knowledge, at least in the form of either concepts or intuitions, presupposes an agent (namely, and agent having this knowledge) in the possession of conscious intentionality. Cf. also footnote 10.
[99] In contrast, a sensation (*Empfindung*) Kant considers subjective. The reason he gives is that a sensation only refers to the subject, viz. "the modification of the subject's state" (see *ibid*.).
[100] See also A165/B206, where Kant appears to say that (pure) intuition forms the object of geometry.
[101] The content of intuitions is something we shall return to later.
[102] Reading proposition where Kant uses *judgment* (*Urteil*).

69

purposes, it is of some future interest to make a couple of clarifying remarks on this point.[103]

In the *Jäsche* logic, Kant explains what, in his view, propositions are:

> A judgment is the representation of the unity of the consciousness of various representations, or the representation of their relation insofar as they constitute a concept (Kant [93], p.597).[104]

Kant explains a proposition as a kind of representation, which, in Kant's view, all items of knowledge are (see above).

Kant acknowledged several types of propositions. Besides categorical propositions, Kant also distinguished hypothetical and disjunctive propositions, among other things.[105] The latter goes beyond a view as expounded by Locke, who only seemed to have recognized categorical propositions (§ 3.1).

A proposition, Kant suggests, is in part constituted by several (other) representations. These representations, in turn, are not merely representations, but items of knowledge instead. This, we think, is confirmed by a characterization of judgment in the *Critique of pure reason*, which, in many respects, seemingly runs along similar lines as the one quoted above. Says Kant: "a judgment is nothing other than the way to bring given cognitions to the *objective* unity of apperception" (B141). In other words, then, a judgment is a way of bringing given *items of knowledge* to the objective unity of apperception.

What could those items of knowledge that in part constitute a proposition be? The currently available options are the following: intuitions, concepts or propositions. In case of the latter, exactly the same question may be asked as before, and the same answer can be given to it. A proposition possibly involved in a proposition may again involve intuitions, concepts, or propositions. If we assume some sort of "principle of well-foundedness," it seems reasonable to conclude that a judgment ultimately involves only intuitions or concepts.

The main point is that, in Kant's view, there are only two prime types of cognition: intuitions and concepts. Kant also recognizes the proposition as an item of knowledge, but only in a derivative sense. In the end, a propositional item of knowledge is constituted by concepts, intuitions, or both.

---

[103] See also A70-6/B95-101, B140-2.

[104] Kant here defines a judgment in terms of a *unity* (of the consciousness of) several presentations. In the *Critique of pure reason*, Kant seems to consider this an improvement with respect to those who hold that a judgment is merely a representation of the relation between two representations (or items of knowledge—see below). For example, Locke may be more or less considered a representative of such a view (§ 3.1). See B140-2.

[105] Besides the ones mentioned in the main text, Kant distinguished other (nine, to be exact) main types of proposition; see A70/B95.

Intuitions: review of the literature. There has been considerable discussion as to how Kant's notion of intuition has to be understood. Given the importance of Kant's notion of intuition for our understanding of his philosophy of mathematics, the discussion is of considerable interest. We will review the debate and point out certain weak spots. We focus on the main contributions, put forward by Hintikka and Parsons.

Much discussion on Kant's notion of intuition has been triggered by what Kant says in the following citation:

> [An intuition] is immediately related to the object and is singular; [a concept] is mediate, by means of a mark, which can be common to several things (A320/B377; cf. also Kant [93], p.589).

Kant says that an intuition relates immediately to the object and is singular. With Parsons, let us say that an intuition satisfies (1) the *immediacy condition* and (2) the *singularity condition* (Parsons [118]).

Parsons thinks that the singularity condition seems clear enough: it means that an intuition "can have only one individual object" (*ibid*., p.44). A concept, Parsons goes on, relates to the objects that fall under it, i.e., the objects in the extension of that concept (*ibid*.). Parsons states provisionally that, thus far, the distinction between intuition and concept corresponds to what we would nowadays refer to as the distinction between individual constant (or singular term) and general term respectively (*ibid*.).

An intuition, Kant says, is related to an (or the[106]) object in a way that is typified as *immediate*. A concept, in contrast, is related to an (or any) object (or thing), "by means of a mark, which can be common to several things." We may say that a concept is *mediately* related to an object. We may illustrate the point as follows. Consider the concept `triangle`. This concept, we may say, relates to an object via a property that can be common among a certain collection of objects. We may reasonably enough surmise that this property is the property TO BE A TRIANGLE.

However, what does it mean to say that an intuition is immediate? It is at this point that controversy begins to arise. In Hintikka's view, an intuition is "simply anything which represents or stands for an individual object" (Hintikka [72], p.130). In this respect, then, Hintikka agrees with Parsons: an intuition satisfies the singularity condition. Furthermore, however, Hintikka has argued that the immediacy condition is in effect noting but an alternative way of stating the singularity condition (Hintikka [74]; see also Hintikka [73], [75]). Consequently, Hintikka claims that

[106] Kant: "[…] bezieht sich unmittelbar auf den Gegenstand […]."

> Kant's notion of intuition is not very far from what we would call a singular term (Hintikka [74], p.45).[107]

Something very close to this is also assumed by Beth [16], [17], [18] (especially chapter 4).

It may not be clear how the immediacy of individual constants can be understood as to coincide effectively with their singularity. Howell [81], p.210-1, has argued that the immediacy of intuitions should be understood in terms of the notion of direct reference as it applies, at least according to some theorists, to individual constants. An individual constant is said to refer directly, roughly if its referent, would it exist, is not determined by the content of that constant.[108] A directly referring individual constant may be thought of as a mere label (or "tag") of its referent (again, would it exist). According to Howell, the immediacy of intuitions has to be understood in analogy with this notion of direct reference. Parsons reports that, in conversation, Hintikka has confirmed that Howell's reading of the immediacy condition (as being analogous to direct reference) was precisely what Hintikka maintained all along (Parsons [118], p.78, n.44).[109] In sum, then, Hintikka holds that intuitions are very close to individual constants, when the latter are understood as directly referring expressions.

We now also have a way of understanding Hintikka when he says that the immediacy condition is in effect nothing but an alternative way of stating the singularity condition. When Hintikka says that an intuition is "simply anything which represents or stands for an individual object" (see above), we may take this as if he claims that an intuition is like a label for an individual object. Consequently, an intuition is at the same immediate in a way as explained above (namely, in terms of an analogy with the notion of direct reference). Let us continue our original discussion.

Parsons disagrees with Hintikka in that the immediacy condition and the singularity condition effectively coincide. According to Parsons, the immediacy of intuitions really adds something over and above their singularity. More specifically, Parsons holds that, for Kant, immediacy "evidently means that the object of an intuition is in some way present to the mind, as in perception" (Parsons [118], p.44). In a later paper, Parsons rephrased his proposal in more phenomenological terms: "immediacy for Kant is direct, phenomenological presence to the mind, as in perception" (Parsons [119], p.66). Note that an analogy with perception remains a constant factor in his understanding of the

---

[107] As suggested several times before, in the present study, we use *individual constant* instead of *singular term*.

[108] Frege, for one, would disagree: for him, the referent of an individual constant, if it exists, is always determined by the content (or *Sinn*, as Frege called it) of that individual constant. Accordingly, reference is mediated by content.

[109] Hintikka indeed often tends to read immediacy as "direct reference to objects" (e.g., Hintikka [75], p.342).

immediacy condition. In Parsons view, the immediacy of an intuition relates it to perception.

Parsons' proposal seems to presuppose a specific philosophical theory of perception. According to this theory, perception is to be understood as *direct* perception. Accordingly, the immediate objects of perception are objects themselves, and not, as for example representationalists would hold, certain representations of those objects (e.g., sense-data) (see, for example, Pitcher [122]).

Let us note that Parsons need not deny outright that intuitions are like individual constants. In particular, he may very well be willing to admit that intuitions are like individual constants in *some* respect, namely, insofar as they satisfy the singularity condition. However, what Parsons would deny is that this is in effect all there is to be said about intuitions, as Hintikka seems to do. In Parsons view, the immediacy condition really adds something new.

On Parsons proposal, it is especially immediacy what makes intuitions genuinely "intuitive." This may already be indicated by the connection Parsons makes between intuitions and perception (see above).[110] Parsons goes on to point out that the intuitive character of intuitions gains plausibility once we realize that for Kant geometry is an applied science. Parsons believes that for Kant, geometry is about physical space and the figures constructed in it (Parsons [118], p.58). (The objects of intuitions, Parsons seems to hold, are precisely those figures in physical space.) The figures in actual space, Parsons seems to believe, appear as objects given to the senses. What a mathematician does is, among other things, to prove theorems about these figures in physical space—or so Parsons holds. It seems, then, when Parsons relates the immediacy of intuition to perception, he quite literally means *sense perception*, and visual perception in particular. Accordingly, in Parsons view, the object of an intuition would be in some way perceived visually.

Parsons views on this score gain some credibility once we realize that Kant himself has made a seemingly intrinsic connection between intuition and sensibility: Kant says that all intuitions are supplied to us only by sensibility (A19/B33).[111] However, other considerations, in contrast, go against Parsons' views. For example, Parsons' belief that mathematician proves theorems concerning objects given to the senses seems to conflict with Kant's own belief that mathematical knowledge (which, for Kant, includes geometry) is a priori (cf. B14). Parsons himself has pointed out that the a prioricity of mathematics is secured by the fact that, at least on some points, a mathematician proves (some of) his theorems in a peculiar way. For Kant, a mathematician would prove

---

[110] Parsons is not fully explicit on this point, but we suspect that he intends to construe the immediacy of intuition in terms of an analogy with the supposed directness of what may be very broadly called *visual* perception. In the light of this, it is perhaps of some interest to add that the Latin root of *intuition* is *intuēri*, which means "to look at."

[111] We will return to this in § 3.3.

73

(some of) his theorems by only taking account of the essential (or definitional) properties of the specific figures that are given to his senses. This would secure Kant's belief that mathematical knowledge is a priori (cf. Parsons [118], p.61).

The point seems besides the mark, however. To say that a mathematician only takes account of the essential properties of a specific figure implies that the knowledge obtained accordingly is *general* instead of merely bearing on that one single figure. It does not seem to imply, however, that this knowledge is therefore a priori. For roughly speaking, knowledge is a priori if it is based only on reason. However, it seems that on Parsons construal, Kant believed mathematical knowledge to be at least partly based on experience. However, we think it doubtful that Kant believed this. Indeed, it is even doubtful whether Kant believed that mathematics involves any experience at all (cf. § 3.3). (See the next section for further discussion of the distinction between a priori and a posteriori.)

In contrast with Parsons, Hintikka denies that Kant's notion of intuition has "intuitivity" built into it:

> In Kant and his immediate predecessors, the term "intuition" did not necessarily have to do with imagination or to direct perceptual evidence. In the form of a paradox, we may perhaps say that the "intuitions" Kant contemplated were not necessarily very intuitive (Hintikka [72], p.130).

Interestingly, Hintikka not only denies a necessary connection between intuition and (direct) perceptual evidence. He also denies a necessary connection between intuition and something he refers to as *imagination* (see also § 3.3). Before we close off this section, let us return once more to Hintikka's belief that Kantian intuitions are very much like individual constants.

Recall Hintikka saying that, for Kant, an intuition is "anything which represents or stands for an individual object as distinguished from general concepts." As a result, Hintikka claimed that an intuition is not far from what we would call an individual term. Taken together, this seems to imply that Hintikka believes that anything that represents or stands for an individual object is or comes close to an individual constant.

This, however, does not seem to be the case. Diagrams may form a serious counterexample in this respect. We may say that a diagram (e.g., any of the diagrams presented in § 1.1) represents or stands for an individual object.[112] However, *prima facie* at least, a diagram does not seem to be an individual constant. In fact, it may even be suggested that a diagram is quite different from an individual constant. For to begin with, the latter is a certain type of linguistic expression, while the former, in contrast, seems to belong to a quite different category (though it need not be clear to *what* category).

---

[112] Though it is not clear that the relation between diagram and object should be or can be understood as being in some way analogous to direct reference.

As it stands, this objection is not very precise. Accordingly, it lacks a clear point. However, even granted its imprecision, it need not even be taken very seriously. What Hintikka surely could deny that the eventual diagrammatic features of intuitions—whatever they precisely are—are *employed* in the course of a proof. On the positive side, it would be primarily with respect to its *role* in the course of a proof that an intuition is not very far from an individual constant. Put differently, it is in the light of the specific *way* mathematicians prove their theorems that intuitions come close to being individual constants. As a matter of fact, Hintikka appears to believe that the role of intuitions in this respect is mainly determined by the rules for the quantifiers in a system of natural deduction.[113] As to its *nature*, an intuition may very well be diagrammatic.[114]

Note that Hintikka's point is a legitimate one. Speaking of individual constants, we are typically inclined to think of a certain category of linguistic expressions, for example, *Plato*, *15*, or *ABC*[115] (as we tended to do in § 2.2). However, from an abstract mathematical point of view, a system of natural deduction on itself does not put many constraints on what it is to *be* an individual constant. Insofar as the present point is concerned, the most significant constraints available within such a system are restrictions on the possible *use* of an individual constant. These restrictions, in turn, are precisely the ones constituted by the introduction rules and the elimination rules for the quantifiers (i.e., $\forall$ and $\exists$).

The Hintikka-Parsons debate formed only the beginning of a long discussion on Kant's notion of intuition. See, for example, Thompson [156], Howell [81], and Brittan [21]. However, we shall lay the matter to rest. In order to close off this section, we notice two weak points in the views discussed thus far. Let us state them.

First, Hintikka thinks that the immediacy condition is an alternative formulation of the singularity condition. This supports a view according to which mathematical reasoning is in many respects like natural deduction. However, this view brings along with it that, for Kant, reasoning in mathematics is a form of propositional reasoning. We shall find reason, however, to reject this. It will turn out that, in Kant's view, reasoning in mathematics is essentially a form of diagrammatic reasoning (§ 4.2).

Second, in contrast with Hintikka, Parsons understands the immediacy of intuitions in such a way that the immediacy of an intuition means that (the object of) that intuition is in some sense immediately present to the mind, as in perception. Accordingly, we may run the risk that mathematical knowledge turns

---

[113] Hintikka [73], p.175, pays particular attention to the elimination rule for the existential quantifier (see § 2.3).

[114] Of course, Hintikka holds that this is not simply Kant's own view. For in that case, Hintikka's point would be a bare anachronism. What he does seem to maintain, however, is that this is a reasonable reconstruction of Kant's views (cf. Hintikka [72], p.131).

[115] For example, in case one refers to a triangle.

75

out to be a posteriori after all, something that runs counter to one of Kant's fundamental beliefs with respect to (pure) mathematical knowledge. We think that Parsons has not correctly understood Kant's claim that intuitions in mathematics are supplied to us by sensibility. This is something that we shall return to later (see especially § 3.3).

Types of intuitions. Contrary to what Hintikka and Parsons seem to assume, Kant's notion of intuition is not one. It turns out that he distinguishes between several types of intuitions. One distinction he makes is that between a posteriori and a priori intuitions. Besides that, he also acknowledged pure intuitions.[116] It is the aim of this section to clarify these distinctions. We begin by considering the distinction between a priori and a posteriori. Next, we turn our attention to pure intuitions.

1. *A posteriori and a priori intuitions*. For Kant, the qualifications *a priori* and *a posteriori* apply to items of knowledge generally, and not only to intuitions (as does the qualification *pure*—see below). It is useful to start from this somewhat broader perspective and thus to consider items of knowledge generally first.

In general, Kant thinks of the qualification a priori, together with its cognate a posteriori, primarily in terms of so-called *sources of knowledge* (cf. B2-3). In other words, the qualifications a priori and a posteriori more or less turn on the question: "where does this or that item of knowledge come from?"

Kant distinguishes two main sources of knowledge (cf. A50/B74). One of these, Kant calls experience (B2). The other source of knowledge he does not seem to give a single separate name, but we may broadly refer to it as *reason*.[117] An item of knowledge is a posteriori if it primarily has its source in experience; an item of knowledge is a priori if it has its source in reason alone. An item of knowledge that is a posteriori Kant alternatively calls *empirical* (B2; cf. B3).

The characterization of the distinction between a priori and a posteriori in terms of sources of knowledge may strike one as somewhat odd. More positively, one may be inclined to think that the distinction between a priori and a posteriori should mark off a distinction between two types of *justification*, and not, as Kant seems to think, between two sources of knowledge. Thus, one may suggest that knowledge is justified a priori roughly if it were justified on the basis of reason only. Knowledge would be justified a posteriori roughly if it were justified on the basis of experience.

Accordingly, there may seem to be some tension between Kant's characterization of the distinction between a priori and a posteriori knowledge and what one would normally understand this distinction. However, from Kant's point of view, the tension is only apparent. For Kant, matters concerning the

[116] Besides intuitions a priori, intuitions a posteriori, and pure intuitions, Kant also distinguished intellectual intuitions (B68).
[117] Kant appears to recognize yet a third source of cognition, namely, apperception; see B132.

origin of an item of knowledge do not stand independently of matters concerning its justification. In particular, Kant seems to have believed that justification is intimately related to the specific procedures that are carried out in order to get an item of knowledge. Thus, for example, the proof described in The Passage is on the one hand a procedure that serves to get a proposition as an item of knowledge (which in this case is a priori). On the other hand, this procedure at the same time justifies this item of knowledge. A similar point applies to items of knowledge a posteriori.

The supposed oddity of Kant's way of understanding the distinction between a priori and a posteriori seems to presuppose a distinction along the lines of the distinction between context of discovery and context of justification (§ 2.4). At any rate, while Kant may have acknowledged some distinction along these lines,[118] we nevertheless think that for Kant matters of discovery and matters of justification where much closer than proponents of this distinction are often inclined to believe.[119] The upshot is that the point does not constitute an objection against Kant in this respect. Rather, it may very well be taken as providing a further qualification of Kant's views. Let us now provide some further elucidation on intuitions a posteriori and intuitions a priori.

A. *Intuitions a posteriori*. When Kant's talks about intuitions and related matters, he tends to appeal to a diverse terminological arsenal. In order to clear things up a bit, imagine the following relatively simple set up, which is certainly not uncongenial to Kant. We have an agent who is engaged with getting items of knowledge. We think of our agent as situated in what we refer to as its "environment." This environment is such that it possibly affects the sensory apparatus of our agent in certain ways.

Now, we may generally say that an item of knowledge has its source in experience—i.e., is a posteriori or empirical—if the way it arises involves the processing of "experiential input," or, as we shall say, appearances (cf. A20/B34). These appearances, in turn, come as the result of certain operations or actions of the environment upon an agent's sensory apparatus. In this respect, Kant characterizes an agent getting an a posteriori item of knowledge primarily as *receptive* (A19/B33; A50/B74). An agent operates certain receptive powers—its sensory faculties. The idea particularly applies to intuitions, yielding intuitions a posteriori, or *empirical intuitions* (cf. B2). Thus, the way an empirical intuition arises involves the processing of appearances that an agent receives as the result of environmental operations upon his sensory apparatus.

However, besides the appearances that an agent receives from its environment, an empirical intuition also involves a certain subjective contribution by that agent himself (B1). Kant explains this in quite elaborate terms in the part of the *Critique of pure reason* called the Transcendental

---

[118] Cf. Popper [126], p.31 (cf. also § 2.4).

[119] This point may very well hold for Kant's contemporaries in general; see also our discussion of Locke's views on proof in § 3.1.

Aesthetic.[120] Here, Kant points out that, in case of empirical intuitions, this subjective contribution specifically concerns the organization of appearances in terms of spatio-temporal relations, that is, in terms of relations in space and/or time (A20/B34; A22/B36). That knowledge accordingly involves a subjective contribution may be considered a typical trait of Kantian philosophy.

Pursuing this line of thought, the aforementioned relations in space and/or time Kant refers to as the *form* of an empirical intuition (cf. A20/B40). The appearances, on the other hand, together constitute what Kant calls the *matter* of that intuition (A50/B74; A723/B751). Alternatively, Kant tends to refer to these appearances as the *content* of that empirical intuition (A59/B83; A723/B751).[121] It becomes suggestive to think of the form of an empirical intuition as a configuration of relations in space and/or time. In the light of this, we may say that an unorganized collection of corresponding (sensory) relata—i.e., appearances—forms the content of that intuition.

The above suggests that the Transcendental Aesthetic primarily concentrates on empirical intuitions. We think that this is indeed the case, as is confirmed by the following. In the Transcendental Aesthetic, Kant clearly indicates that the way experience (i.e., appearances) comes about is pretty much a passive (or receptive) affair. For example, Kant says: "[t]he effect of an object on our capacity for representation, insofar as we are affected by the object, is *experience*" (A19-20/B34). In terms of our explanatory set-up presented earlier, this clearly suggests that experience arises from operations of the environment on an agent's sensory apparatus. Therefore, the intuition that arises accordingly must be empirical (cf. A20/B34; cf. also above).

Since the form of an empirical intuition resides in the subject, what makes an empirical intuition genuinely empirical seems to bear on its content and not so much on its form. This point can be linked to the idea of sources of knowledge introduced above. We may say that, insofar as an empirical intuition (which, in Kant's view, is an item of knowledge of a sort) is concerned, its source does strictly speaking not concern that intuition *per se* but much more its content. Thus, it is primarily the content of an empirical intuition that comes from experience and not the form of that intuition.

---

[120] In the second edition, this corresponds to B33-73.

[121] Kant says:

> […] there are two components to the appearance through which all objects are given to us: the form of intuition (space and time), which can be cognized and determined completely *a priori*, and the matter (the physical), or the content, which signifies a something encountered in space and time […].

We think that Kant's parenthetical addition "the physical" makes clear that Kant is mainly thinking in terms of empirical intuitions here. There appears nothing, however, that should prevent us from applying the form-content distinction also to intuitions a priori. We shall return to this later.

An empirical intuition, Kant says, is one "which is related to the object through sensation" (A20/B34). Very likely, this object is an individual object. What we see, then, is that an empirical intuition satisfies the singularity condition. This agrees with what Hintikka and Parsons claimed about intuitions generally (see § 3.2).

Having mentioned Hintikka again, observe also the following: the short quote given in the previous paragraph may suggest that an empirical intuition does not refer "directly" (as we may put it) to its object. Again, Kant says that it refers to its object *through experience*, which appears to give the reference a kind of "indirect" flavor. Perhaps we may say that the reference of an intuition a posteriori is mediated by the sensory processes giving rise to its content.

At any rate, if this conclusion would hold, then the way Hintikka (and Howell) understand the immediacy of intuitions seems to become particularly problematic. For Hintikka held that intuitions referred to their object in a way that can be typified as "direct" (i.e., in a way that is analogous to direct reference in case of individual constants). Thanks to this, he could subsequently say that intuitions are in effect very close to what we call individual constants (see above). However, it may very well be that this comes down to a misunderstanding. We have found some indication to think that Kant thought intuitions not to refer to their objects in a way that can be typified as "direct." Accordingly, the analogy of intuitions with (directly referring) individual constants seems to break down on this point. Let us make one further remark on behalf of the short quote that prompted these considerations.

It seems very difficult to explain *how* the spatio-temporal organization of content—i.e., appearances—can bring about a reference to an object. We do not want to answer this question here since this obviously brings us beyond the scope of this study. Furthermore, we think that there are also other problems surrounding the objects that are associated with empirical intuitions. These problems at least concern their nature and their ontological status. Furthermore, the nature of the relation between an empirical intuition and its object is likewise far from clear. Issues like these seemingly touch upon very fundamental questions, such as "what, in Kant's view, is the nature of representation?"[122]

We think it nevertheless plausible to say that an empirical intuition can be typically associated with, or seen as arising from, something like an occurrence or episode of visual perception of an object. Insofar as Kant's broader philosophical interests are concerned, one can easily imagine that empirical intuitions would play their role in what we may roughly refer to as the empirical sciences.

If the aforementioned occurrence or episode of visual perception can be characterized as being akin to something like "direct perception" (as Parsons seems to do; cf. § 3.2) then we find that empirical intuitions also satisfy the

[122] Remember that Kant considered an intuition as a kind of representation.

 79

immediacy condition. In this respect, we tend to go with Parsons, but only in the case of empirical intuitions. How things stand in case of intuitions a priori is as yet undecided (see below). Let us now turn to intuitions a priori, which play a substantial role in Kant's view on the mathematical method, and the mathematical method of proof in particular (cf. § 3.3).

B. *Intuitions a priori*. Generally, if an item of knowledge has its source in reason—i.e., is a priori—, then the way it arises does not involve the processing of appearances that come as the result of environmental operations on an agent's sensory apparatus. In contrast, this item of knowledge now arises independently of the sensory input he receives from his environment (cf. B2-3). More positively, an item of knowledge a priori is one that arises from reason alone. This particularly holds for intuitions a priori: an intuition a priori has its source merely in reason (see below for a possible qualification).

Recall the form-content distinction as it applies to empirical intuitions. The line of thought presented there was essentially the line of thought offered by Kant in the Transcendental Aesthetic. We argued that this line of thought concentrated primarily on empirical intuitions. That said, we do not mean to say, however, that it is simply out of the question that the considerations put forward in the Transcendental Aesthetic also apply to intuitions *a priori*. In particular, we do not mean to say that Kant had no idea of a distinction between form and content in the case of intuitions a priori. Nevertheless, considering what Kant says in the Transcendental Aesthetic, it appears somewhat hard to see how, exactly, this distinction is to be understood. The trouble is that the content of an intuition a priori cannot arise from experience. In other words, the content of an intuition a priori does not consist of appearances that come as the result of environmental operations on an agent's sensory apparatus. However, if the content of an intuition a priori does not arise from experience, from what else *does* it arise? We postpone this question to § 3.3.

Related to the previous paragraph, note also the following. If the content of an intuition a priori does not come from experience in the way indicated, then we would expect that intuitions a priori do not refer to their object though experience (as empirical intuitions apparently do; see above).[123] We would expect that they refer to their object through something else instead. As a result, it seems to become difficult to compare an intuition a priori with an occurrence or episode of visual perception, as Parsons does (see above). This, then, is yet another point on which an intuition a priori differs from an empirical intuition.[124]

We can make two further points on behalf of intuitions a priori that were also made in case of empirical intuitions. First, an intuition a priori no doubt satisfies the singularity condition: it refers to an individual object. Second, for the similar reasons that where brought up in case of empirical intuitions, it seems

[123] As to this object, the same difficulties apply as in the case of empirical intuitions (cf. above).

[124] The other being that the respective contents of empirical intuitions and intuitions a priori have a different source.

problematic to say that intuitions a priori satisfy the immediacy condition as understood by Hintikka (and Howell). As in the case of empirical intuitions, there is some reason to believe that the reference of intuitions a priori, too, has an "indirect" or mediate flavor. In particular, we suggested that the reference to an object in case of intuitions a posteriori may very well be mediated by certain sensory processes. However, what would the nature of the processes be that mediate reference in case of intuitions a priori? We return to this question in § 3.3.

2. *Pure intuitions*. Kant often discusses empirical intuitions in tandem with what he calls *pure* intuitions (e.g., A19-20/B34-5; A42/B59-60; A50/B74). He sometimes characterizes *pure* intuitions—or indeed, pure items of knowledge generally—rather negatively as intuitions that are not mixed up with anything that stems from experience (cf. A20/B34; A50/B74). Reasonably enough, then, pure intuitions are not empirical intuitions.

On other occasions, Kant is more extensive in his explanations concerning pure intuitions and accordingly gives us some positive information. This is particularly the case in the Transcendental Aesthetic. One line of thought presented there we have met earlier. Let us nevertheless repeat it here.

Recall that in case of an empirical intuition, we can distinguish between, on the one hand, the appearances that come as the result of certain environmental operations on an agent's sensory apparatus. On the other hand, there is a certain subjective contribution by that agent himself. This subjective contribution turned on the organization of the appearances in terms of relations in space and time. The former—the appearances—Kant referred to as the content (or matter) of an empirical intuition; the latter as its form. (See above.)

Remarkably, however, Kant goes on to refer to the form of an empirical intuition itself as an intuition of a sort. He calls it a *pure* intuition (A20/B34-5). It would seem, then, that at least insofar as the Transcendental Aesthetic is concerned, pure intuitions could be seen as forming a kind of reified formal aspect or feature of empirical intuitions.[125] This formal feature of an empirical intuition precisely consists in the spatio-temporal structure of its content.

Pure intuitions appear to have a somewhat ambiguous status in Kant's thought. One the one hand, Kant introduces them in terms of a formal feature of empirical intuitions (in the Transcendental Aesthetic). On the other hand, he apparently considers them as intuitions, and hence as items of knowledge (§ 3.2). Consequently, we would expect that pure intuitions are not mere form; as items of knowledge, they would be in the possession of content instead.

Rather than in terms of form alone, it indeed appears that Kant sometimes tends to think of pure intuitions as a configuration of relations in space and time,

[125] Pure intuition, Kant holds, can be examined apart from the matter of an empirical intuition (A20/B34). This is precisely what Kant does the Transcendental Aesthetic (cf. A21/B35-6).

but seemingly *with* content. For example, at one point, Kant explains how to judge synthetically concerning a concept, either a priori or a posteriori:

> […] I can go from the concept to the pure or empirical intuition corresponding to it in order to assess it *in concreto* and cognize *a priori* or *a posteriori* what pertains to its object (A721/B749).

The way Kant puts pure intuitions on a par with empirical intuitions suggests that he considers them similar in many respects. Though the case is not entirely clear, we may particularly suspect, then, that, at least on this occasion, pure intuitions have content just as empirical intuitions do (though their respective contents may arise from a different source). (See, e.g., also A719/B747; A722/B750).

It is perhaps of some interest to add that the above quote is taken from the part of the *Critique of pure reason* where Kant, among other things, outlines his views on the methodology of mathematics.[126] It is especially there where Kant occasionally, but not always, seems to think of pure intuitions not as pure form but as items of knowledge with content instead. Even more so, in this part of the *Critique of pure reason*, Kant sometimes seemingly tends to identify pure intuitions with intuitions a priori. For example, when he says: "off all intuition none is given *a priori* except the mere form of appearances, [i.e.,] space and time" (A720/B748; cf. also A724/B752).

However, Kant suggests that an item of knowledge a priori is in general not pure. In fact, Kant provides us with an explicit counterexample (cf. B3). Consider the proposition that every change has its cause (which Kant takes as an item of knowledge). In Kant's view, this proposition is a priori, though it is not pure. The reason Kant offers is that the concept `change` involved in it is an item of knowledge "that can be drawn only from experience."

Note also the following. Reasonably enough, the concept `change` is, in Kant's view, not a priori.[127] What the example suggests is that though a proposition may be a priori, the concepts involved in it need not. In Kant's view, the proposition that every change has its cause arises from reason alone. This does not hold, however, for all the concepts involved in it. Consequently, this example suggests that, in Kant's view, discovery and justification do not always go entirely hand in hand. While an item of knowledge may be justified a priori, certain elements involved in it may still arise from experience. Let us return to intuitions.

What we have found is that if item of knowledge is a priori, it is not always pure. A natural question now is the following: is every pure item of knowledge a priori? As may be expected, Kant's answer to this question is affirmative (see B2; cf. also A20/B34-5). When we restrict ourselves to (pure) mathematics, all

[126] This part is titled: The Discipline of Pure Reason in its Dogmatic Use; see A712-38/B740-66.
[127] Presumably, it is a posteriori.

items of knowledge are pure, and hence a priori (cf. A713/B741, A718/B746). We conclude that, insofar as mathematics is concerned, “a prioricity” and “pureness” coincide: in (pure) mathematics, every item of knowledge a priori is pure and *vice versa*. This conclusion holds for intuitions in mathematics in particular. Thus, in mathematics, an intuition a priori is pure and *vice versa*. Whether they should be distinguished outside mathematics is a question we shall not discuss here.

Summary and conclusions. Let us recap the discussion thus far. Kant distinguished at least three types of intuitions: intuitions a priori, empirical intuitions (or intuitions a posteriori), and pure intuitions. Insofar as intuitions in pure mathematics are concerned, intuitions a priori coincide with pure intuitions. An intuition a priori is one whose source lies in reason alone. An empirical intuition, in contrast, is one whose source lies in experience. More specifically, the content of an intuition a priori arises independently from the environmental actions on an agent’s sensory apparatus. It merely arises by exercising certain powers that have to be associated with reason alone (see the next section). The content of an empirical intuition, in contrast, arises as the result of the processing of sensations that arise from environmental actions on an agent’s sensory apparatus. It appears, then, that intuitions must not be considered independently of the specific processes delivering these intuitions as their respective products. In case of intuitions a priori, these processes have to be associated with reason; in case of empirical intuitions, these processes have to be associated with sense perception. This may very well be a point that Hintikka did not realize. Parsons, in contrast, may have been aware of something along these lines when he connected the intuitiveness of intuitions with sense perception (and hence perceptual processes). However, in the next section, we shall see that Parsons point is in need of adjustment.

§ 3.3. Kantian construction

At the beginning of the previous section we quoted Philip Kitcher saying that, insofar as the philosophy of mathematics is concerned, the term *intuition* suffers from being overworked. A similar point could certainly be made on behalf of *construction*.[128] As is the case with intuition, this puts heavy demands on our explanations turning on Kant’s notion of construction.

[128] Hesseling [67], chapter 4, discusses several conceptions of construction in the foundational debate during the first decades of the twentieth century. He also includes a short history of different notions of construction that have been in the air since antiquity (*ibid*., pp.108-11). In the light of this, “constructivism” is not a label for a group of people who all have more or less the same philosophical views with respect to mathematics. In contrast, there are various different forms of constructivism. See also Detlefsen [34] for a discussion.

83

The purpose of this section is to come to grips with Kant's notion of construction. We begin by pointing out that Kant saw construction primarily as a methodical feature of mathematics, and the method of proof in mathematics in particular. Next, we clarify Kant's idea that construction is an act or process of exhibition *a priori*.

Construction as a feature of the mathematical method. Construction, Kant appears to hold, primarily applies to concepts—it is concepts that are constructed. Kant considers the construction of concepts as an essential methodical means in mathematics:

> The essential feature of pure *mathematical* cognition, differentiating it from all other *a priori* cognition, is that it must throughout proceed *not from concepts*, but always and only through the construction of concepts […] (Kant [95], p.65; A713-4/B741-2).

The construction of concepts is something that concerns the *procedure* (or method) of mathematics (A718/B746, A726/B754).[129] In Kant's view, the method followed by a mathematician to the end of proving theorems is constructive: a mathematician proving a theorem essentially constructs concepts. Even more so, Kant holds that without constructing concepts, a mathematician cannot take a single step forward (cf. Kant [95], p.77). In particular, without construction, a proof would not get "off the ground," so to speak.

Construction first and foremost concerns the cognitive procedure or method a mathematician executes when he proves a theorem. Furthermore, based on what Kant says in the above citation, it seems to follow that if a procedure would not proceed by means of the construction of concepts, it would not be a *mathematical* procedure. In contrast, it would seem to be another type of procedure at best. Therefore, it is part of Kant's views on proof that mathematical proof is a distinctly *mathematical* procedure, as opposed to other types of proof procedures (see § 5.2). It is precisely construction forming the distinguishing feature of mathematical proof. A mathematician, Kant sometimes says, reasons *demonstratively* (or *intuitively*). To reason demonstratively means to reason by means of the construction of concepts (cf. A735/B763). Alternatively, Kant also says that a mathematician reasons *in concreto*, i.e., by means of the construction of concepts in terms of intuitions (cf. A734/B762; see also below).[130]

The upshot of this section is a short one indeed: Kant claimed that a mathematician, *qua* mathematician, essentially proves his theorems by means of constructing concepts *in concreto*. Let us try to come to closer grips with this idea.

---

[129] Kant's term is *Verfahren*.

[130] A philosopher, in contrast, would reason *in abstracto* (A734/B762). Alternatively, Kant says that a philosopher reasons *acroamatively* instead of demonstratively (A735/B763).

Construction as exhibition a priori. What Kant means by construction of concepts in mathematics he explains as follows:

> […] mathematical cognition is [*rational cognition*] from the *construction* of concepts. But to *construct* a concept means to exhibit *a priori* the intuition corresponding to it (A713/B741).

Construction in mathematics, as Kant explains it, is the construction of a *concept*.[131] Construction in mathematics is a process of exhibition, which is qualified as being *a priori*. That what is exhibited is an intuition. This intuition, in turn, in some sense corresponds to the concept that is constructed accordingly.

Note that Kant does not seem to qualify the intuition exhibited as a priori but rather the exhibitive procedure. This may seem strange in view of things said in § 3.2. What we found there was intuitions admitting of the qualification a priori. Recall, however, that we also noticed that the a prioricity of an intuition hardly stands independent of the processes producing that intuition. It is reasonable to assume that we have now found a point in case: an intuition that is exhibited a priori is an intuition a priori and *vice versa*.[132]

What does it mean—to *exhibit* an intuition? We have found it hard to find a place where Kant offers some useful and elaborate explanation on this point. Except, perhaps, where Kant says that "[t]o be given an object […] is […] to be exhibited immediately in intuition" (A156/B195). We may perhaps say that to exhibit an intuition means to give an object to that intuition. This point can be pursued a little bit further.

Remember that we found indication to believe that, in Kant's view, an intuition (be it empirical or a priori) refers to its object through its content. Also, we noticed that Kant manifests a general tendency to conflate the object of an item of knowledge with its content, a point that hence applies to intuitions (§ 3.2). We may perhaps conclude that to exhibit an intuition means to provide that intuition with content. In other words, then, to exhibit an intuition means providing the relations (i.e., the relations in space an time) in part constituting that intuition with content. However, where this content comes from?

Kant says that in mathematics, an intuition is exhibited *a priori*. We think it is the qualification *a priori* providing the clue to the aforementioned question. To

---

[131] The opening sentence of The Passage ("He begins at once to construct a triangle.") does not necessarily conflict with this. One may point out that construction here is primarily the construction of an object (i.e., an individual triangle). Note, however, that an inclination to think so may very well be merely caused by Kant's way of speaking. Compare: we naturally speak of the definition of a triangle. What is typically meant, however, is not the definition of an object (i.e., an individual triangle) but rather of a concept (or a term) (cf. also A718-9/B746-7).

[132] The German text, however, goes both ways: "Einen Begriff aber konstruieren heißt: die ihm korrespondierende Anschauung a priori darstellen." Here, the qualification *a priori* may apply to the intuitions, the exhibitive procedures, or both.

exhibit an intuition a priori means to provide an intuition with a content independently of all the sensations resulting from the environmental operations on an agent's sensory apparatus. Looking ahead to things yet to come, we suggest that it is mainly on this point where many readings of Kant's notion of intuition in mathematics go wrong. However, to say that the content of an intuition arises independently of environmental actions on an agent's sensory apparatus merely amounts to a negative characterization of the 'a priori' in *exhibition a priori*. We postpone the issue to the next section.

It is natural to wonder whether Kant also acknowledges exhibition *a posteriori*. As far as we can see, Kant does not literally speak in these terms. However, we can easily imagine that he would have acknowledged it. In the light of things said in the previous paragraph, we may propose that to exhibit an intuition a posteriori means to provide an intuition with a content, one that comes as the result of environmental operations on an agent's sensory apparatus. We would expect that the product delivered a running a procedure of exhibition a posteriori is an empirical intuition. Indeed, the content of an empirical intuition comes from experience (§ 3.2). Perhaps, then, we may compare a procedure of exhibition a posteriori with a perceptual procedure. Running such a procedure yields something that can perhaps be compared with something like perceptual state or episode.

The productive imagination. We shall now try to answer a question that remained open in the previous section: if not from experience, where *does* the content of an intuition a priori in mathematics come from?

As suggested in the previous section, this forms an issue that has puzzled many commentators on Kant's philosophy of mathematics. In particular, many scholars concerned with Kant's philosophy of mathematics have found trouble in Kant's suggestion that all intuitions are sensible (Parsons [118], Hintikka [73]). As observed earlier, Kant says that only sensibility supplies us with intuitions (A19/B33). Part of the problem lies herein that Kant is not always very clear about what he precisely means by *sensibility*.

We argued that (§ 3.2), in the Transcendental Aesthetic, Kant appears to associate sensibility primarily with a certain *receptivity* of an agent, namely, a receptivity for being affected by environmental operations on an agent's sensory apparatus. It strikes us as if many Kant commentators have understood Kant's notion of sensibility primarily in these receptive terms, by mainly concentrating on the Transcendental Aesthetic. Note, however, that accordingly, intuitions are effectively identified with intuitions a posteriori. As a result, it becomes hard to make sense of intuitions in mathematics, except if we identify them with intuitions a posteriori (as, for example, Parsons seems to have done; cf. § 3.2). It should be clear, however, that the intuitions that play their distinct role in pure mathematics are, in Kant's view intuitions *a priori*. It seems that the crux of the difficulty turns on the respective sources of an intuition.

An important point that needs to be made is that Kant not only relates sensibility to sense perception (as he mainly seems to do in the Transcendental Aesthetic) but also to what he broadly refers to as the *imagination*: "[…] the imagination […] belongs to *sensibility*" (B151[133]). In general, Kant characterizes the imagination in general as "as the faculty for representing an object even *without its presence* in intuition" (*ibid.*). In this respect, the imagination contrasts with sense perception, by means of which objects can be represented that *are* present. It strikes us as if most Kant commentators have overlooked precisely this point: that Kant associates sensibility not only with sense perception, but *also* with the imagination.

In the previous paragraph, one difference between sense perception and the imagination was mentioned. However, there is another and related way in which the imagination differs from sense perception. Thus, Kant says that

> its [i.e., the imagination's] synthesis is still an exercise of spontaneity, which is determining and not, like sense, merely determinable (B151-2).

Kant thinks that the productive imagination is *spontaneous* (but see below for a qualification). It is especially the spontaneity of the imagination occasionally making Kant's use of the term *sensibility* somewhat misleading. For, as we have seen, Kant often tends to typify sensibility primarily in receptive terms (especially in the Transcendental Aesthetic). However, we can now conclude that this is only part of the story: in the form of imagination, sensibility is spontaneous.

Upon closer inspection, Kant's notion of the imagination is quite subtle. In particular, Kant succeeded in making a couple of pertinent distinctions:

> Now insofar as the imagination is spontaneity, I also occasionally call it the *productive* imagination, and thereby distinguish it from the *reproductive* imagination, whose synthesis is subject solely to empirical laws, namely those of association of association, and that therefore contributes nothing to the explanation of the possibility cognition *a priori*, and on that account belongs not in transcendental philosophy but in psychology […] (B152).

Kant explains the distinction between productive and reproductive imagination in more elaborate terms in the *Anthropology*:

> The imagination (*facultas imaginandi*), as a power to intuit even when the object is not present, is either *productive* or *reproductive*. As productive, it is a power of original exhibition of the object (*exhibitio originaria*), and hence of an exhibition that precedes experience. As reproductive, it is a power of derivative exhibition (*exhibitio derivativa*), an exhibition that

---

133 This citation is not from the Transcendental Aesthetic, but from that part of the *Critique of pure reason* called the Transcendental Logic.

87

> brings back to the mind an empirical intuition that we have had before (Kant [89], p.p.167; cf. also Kant [96], p.240).

Kant makes a distinction between the productive and the reproductive imagination. What we now see is that the imagination is not a spontaneous (rather than receptive) power *per se*. Spontaneity primarily applies to the *productive imagination*. Furthermore, note that, in Kant's view, the reproductive imagination contributes nothing to the explanation of the possibility of a priori cognition. (Kant adds that the reproductive imagination is solely subject to empirical laws and the laws of association in particular.) Therefore, if the imagination is of any relevance for Kant's account of mathematical knowledge, it must be in the form of productive imagination.

The productive imagination, Kant says, is a power of *original* exhibition (cf. above). It is a power that enables an agent to exhibit an object in a way that the object exhibited need not be experienced first. Accordingly, we may say that the productive imagination is, in contrast with the reproductive imagination, properly *creative* in this respect.[134] Furthermore, since exhibition by way of the productive imagination precedes experience, we see no other way but concluding that exhibition by way of and exercise of the productive imagination is in fact exhibition *a priori*. The latter leads us to construction in mathematics.

Where does this all lead to in view of the question posed at the beginning of this section? We think that, in Kant's view, there are no other ways of exhibiting intuitions—i.e., providing them with content—except by way of experience or by way of the imagination. Since in the former case only empirical intuitions can arise, we are led to conclude that the content of intuitions a priori comes as the result of exercising the productive imagination. Hence, an intuition a priori in mathematics refers to its object not through experience, but through the productive imagination instead.

We can now see why it is so difficult to understand the immediacy of intuitions in general with the supposed immediacy of a state or episode of visual perception, as Parsons did (cf. § 3.2). While such a comparison seems to make sense in case of empirical intuitions, it does not work in the case of intuitions a priori. From Kant's point of view, we may say that visual perception involves an exercise of an agent's receptive powers. More specifically, it involves the processing of the sensations that arise as the result of environmental operations on his sensory (i.e., visual) apparatus. However, in the case of an intuition a priori, such environmental operations do not play any significant role: the way an

[134] Another way of understanding the productivity of the productive imagination, Kant says, is in terms of the *voluntary* production of imaginings. The voluntary production of imaginings Kant calls *fantasy* (which he relates to fiction in turn; Kant [89], pp.167-8). However, Kant goes on that this does not mean that the productive imagination, as understood accordingly, is properly creative. In contrast, Kant holds that in case of fantasy, we can always show from where the (productive) imagination took its content (cf. *ibid*.).

intuition a priori arises is entirely independent of such operations. An intuition a priori involves the exercise of a spontaneous power, namely the productive imagination.

As a result, we see that construction in mathematics is an act of the productive imagination. In particular, construction, in Kant's view, should not be understood in terms of mechanical construction by means of, for example, ruler and compass (the ruler is an unmarked straight edge). Of course, in case of traditional elementary geometry, many Kantian constructions are also constructible by means of ruler and compass. However, not all of the Kantian constructions are mechanically constructible, as is indicated by the following.

A *polygon* is a closed plane figure with n sides ($n \geq 3$). In particular, we call a 3-*gon*, a closed plane figure with 3 sides, a 4-*gon* a closed plane figure with 4 sides, and so on. A polygon is called *regular* if all its sides have the same length and if all its angles are of the same size.

Now, it is known that, for example, a regular 9-gon is not constructible by means of ruler and compass (cf. Stewart [152], p.58).[135] However, it would seem that (the concept of) a regular 9-gon can be exhibited, for example, by imagining one. Accordingly, it would seem to be constructible in the sense of Kant.[136]

It appears that Kantian construction should be mainly characterized in terms of what may be called *imaging*. Kantian construction does not primarily take place by mechanical means such as ruler and compass. In contrast, Kantian construction first and foremost takes place by means of exercising a particular mental power: the productive imagination.

Collectively, we conclude that the immediacy of intuitions a priori cannot be construed in terms of the supposed immediacy of a state or episode of immediate visual perception of an object. The main reason is that intuitions a priori do not turn on objects that come "from the outside," so to speak. It is the mathematician himself, in contrast, who produces intuitions a priori. In mathematics, intuitions a priori come as the result of running a creative procedure that is driven by the productive imagination.

The upshot of this section is a striking one indeed: in Kant's view, the source of mathematical knowledge is the productive imagination. Especially today, the imagination is typically associated with literary or other forms of artistic creation.

---

[135] A theorem due to Gauss (dated 1796) says that a sufficient condition for regular n-gon to be constructible is that n is of the form $2^k p_1 p_2 \cdots p_r$, where k is a nonnegative integer and the p_i ($1 \leq i \leq r$) are distinct Fermat primes. A *Fermat prime* is a prime number of the form

$$F_n = 2^{2^n} + 1,$$

where $n \geq 0$ is an integer. (In 1836, Wantzel proved that this condition is also necessary.)

[136] That construction in the sense of Kant goes beyond mechanical construction is further confirmed by Kant's point that constructions also play an essential role in algebra. In Kant's view, construction in algebra is more specifically referred to as *symbolic construction*. The latter, in turn, can be understood in terms of the creation of notations. See A717/B745.

89

Accordingly, one finds hardly any connection between matters pertaining to the imagination on the one hand and matters of rationality and scientific knowledge on the other.[137] However, for Kant, the connection between the imagination and scientific (or rational) knowledge is an intimate one indeed. This particularly holds for mathematics, which is traditionally often taken to form the paradigm type of scientific knowledge. Construction in the sense of Kant (i.e., as an exercise of the productive imagination) is a rational procedure yielding rational knowledge (A724/B752; cf. also A713/B741).

As a final point, recall from § 3.2 that Parsons attempted to come to grips with the "intuitiveness" of intuitions by making a connection with sense perception. We can now say that this proposal is in general not correct and needs to be adjusted. Although Parsons may be right in case of empirical intuitions, he point does not hold in the case of intuitions a priori. The intuitiveness of intuitions a priori should be understood in terms of their connection with the productive imagination.

§ 3.4. The diagrammatic structure of intuitions

The aim of the present section is to argue that intuitions are diagrammatic. Since Kant's own remarks do not give us enough footholds, we will not hesitate to resort to recent insights on the nature and structure of diagrams. Accordingly, the present section provides a reconstruction of Kant's views.

Intuitions as diagrams. As we have seen (§ 3.2), an intuition is in general constituted by, on the one hand, a certain configuration of relations in space and time. On the other hand, an intuition also involves an unorganized collection relata (the content of that intuition). The relations locate the relata relative to one another in space and time:

> Whatever in our cognition belongs to intuition contains nothing but mere relations: of places in intuition (extension), of change of places (motion) […] (A49/B66-7; cf. A22-3/B37; cf. also § 3.2).

Consequently, intuitions are themselves fundamentally characterized by their being in space and time. This does not imply that intuitions are physical particulars. Kant sometimes suggests that he thinks of intuitions in terms of shapes and durations (A21/B35; A141/B180). More specifically, Kant sometimes suggests that an intuition can be seen as a shape developing itself in time (cf. A724/B752). Shapes and durations, in turn, are in some sense abstract objects.

---

[137] Recently, however, Heal [64] has given a prominent place to the imagination in her analysis of knowledge of other minds.

Note also the following. A typically modern complaint against Kant is that he could not offer an account of reasoning with relations. Reasoning in geometry, however, employs relations very often, witness examples such as "is parallel with," "lies adjacent to," "lies to the left of," etc. Such an account of relational reasoning would only be possible since the theory of relations was developed around the turn from the nineteenth to the twentieth century. This point has been stressed by Russell [137], pp.457-8, and was repeated many times afterwards.

We now see, however, that in Kant's views, intuitions are to considerable extent constituted by relations. This observation may very well put Russell's complaint against Kant in an entirely different light. If the relations constitutive for an intuition are employed in the course of a proof (and we think that they are; see § 4.2), then, contrary to what Russell says, then we possibly have here the beginnings of a Kantian account of reasoning with relations. Whatever the details of such an account would look like, the least we can say is this. In Kant's view, relations would not be expressed in terms of relational expressions such as *is parallel with*, *lies adjacent to*, *lies to the left of*, etc. (as Russell would have it), but intuitively, i.e., in terms of intuitions.

Of course, relational reasoning, as we nowadays see it, heavily involves the use of variables and quantifiers, and it is far from clear how Kant would take account of *that*. Nevertheless, it clearly appears that Kant does at least have a means to represent relations. This seems to give reasoning with intuitions—if there is such a thing—a potential expressive richness going significantly beyond the syllogistic logic forming the standard in Kant's time. For syllogistic logic, due to its "monadic" nature, cannot take account of reasoning with relations. It is interesting to wonder to what extent Kant himself was aware of this potential richness of intuitions with respect to standard syllogistic logic. Let us return to our original discussion.

We noticed that, in Kant's view, an intuition is to a considerable extent constituted by spatio-temporal relations—relations in space and time, that is. In what follows, we mainly consider intuitions insofar as they are constituted by relations in space. In Kant's view, temporal relations typically come into play in case of inferences involving, for example, continuity (cf. A162-3/B203-4; A168-71/B210-3). However, entering into this vast territory would quickly make this work too lengthy. See Friedman [49], chapter 1, and especially pp.71-80, for a historical discussion of various issues involved.

Thus, as we think of them, intuitions can be seen as content organized in space (in terms of spatial relations). This suggests that intuitions are items of knowledge of a quite specific format. We submit that intuitions are in this respect diagrammatic. In order to come to closer grips with this proposal, Kant himself does give us enough footholds. Therefore, we set our discussion of Kant on a hold for a while, and briefly turn our attention to more recent discussions of diagrammatic representations and diagrammatic reasoning. Thus, we hope to find convenient and illustrative concepts serving us as a means in order to deepen our understanding of Kant's notion of intuition.

Using modern resources in order to increase understanding of Kant's views is not uncommon in contemporary Kant scholarship. For example, people have often found it profitable to resort to modern, logical means as an instrument in order to interpret Kant. Hintikka [72], [73], [75] is a notable example in this respect. Parsons also holds that "one must use what one knows" in order to gain understanding of a philosopher, and modern logic need to form no exception in this respect (cf. Parsons [118], pp.43-4). However, rather than logic we think it is also quite useful—appropriate, indeed—to turn our eyes towards certain insights arising from disciplines such as cognitive science and AI. And this is precisely what we shall do. Kant's views on reasoning and proof raise themes that, from a contemporary point of view, not only pertain to logic, but also have close affiliations with problems and issues discussed in these fields (cf. § 2.4). Let us briefly review the relevant literature in order to see what main characteristics are attributed to diagrams.

Though there is no general agreement on how to characterize a diagram (Jamnik [85], p.3), many scholars are of the opinion that diagrams are *spatial* representations, that is, diagrams are representations *in space*. See, for example, Glasgow and Papadias [57], p.356; Chandrasekaran [28], p.2. In this respect, diagrams are thought to contrast with propositions or sentences. For example, a written sentence is written in some space (e.g., the flat, two-dimensional surface of a piece of paper). However, the spatial arrangement of the constituents of that sentence does not seem to be constitutive for that sentence. We may say that the spatial structure is a feature of the medium in which the sentence in realized, and not of the sentence itself. In a way, the medium constrains the sentence to be structured in a certain spatial way.

The point may be strengthened by noting that the same sentence may be spoken. Accordingly, the constituents of that sentence are primarily arranged in time, and not so much in space. By the same token, neither the temporal structure of a spoken sentence is constitutive for that sentence. In contrast, the temporal structure of a spoken sentence is a feature of the medium in which that sentence is realized as a spoken sentence. Perhaps, then, we better not speak of a *written sentence*, but of "a sentence-as-it-is-written" instead. A similar point holds for spoken sentences.[138]

A visual aspect is often thought to form another characteristic feature of diagrams. Diagrams are supposed to be visually perceivable. Again, see Glasgow and Papadias [57], p.356; Chandrasekaran [28], p.2. Kulpa has argued that the visual aspect of diagrams implies that diagrams are at least two-dimensional (Kulpa [98], p.81). We return to this point later.

[138] This distinction between sentences on the one hand and the media in which they are realized on the other is adapted from Lyons [108], p.60. Below, we apply the distinction to diagrams.

Note that insofar as diagrams are visual, it becomes problematic to think of Kant's intuitions a priori as diagrams. However, a caveat is in order here. For it appears that the cognitive science and AI literature on diagrammatic reasoning has its focus almost exclusively on what are often called *external diagrams*. These are defined as certain artifacts printed (or written, or drawn) on, for example, a piece of paper or on a computer screen. Agents in the possession of the appropriate faculties can visually perceive external diagrams. External diagrams contrast with internal diagrams, which are diagrammatic representations in the mind. Internal diagrams are also called *images* (cf. Glasgow, Chandrasekaran, Narayanan [58], introduction, p.xvii)).

It does not seem, however, that intuitions a priori can be compared with external diagrams, precisely because the latter are visible (cf. § 3.3). However, we should note the following. Suppose a mathematician proving a theorem employs an intuition a priori. Typically, this mathematician will have his proof accompanied by a diagram he draws on, say, a piece of paper (or on the blackboard). The latter is an external diagram. We may suspect, that the structure of this external diagram supplies us interesting information about the structure of the intuition a priori which plays an essential role in the proof. Perhaps, then, a consideration of external diagrams will yield us an interesting inside view on Kantian intuitions a priori.

Can intuitions a priori be compared with internal diagrams? This is a question difficult to answer. One consideration that seems relevant here is that, in Kant's view, intuitions are representations whose objects live at the conscious level (§ 3.2). However, it is not clear that the object of an internal diagram is always something an agent is conscious of. Though admittedly, our impression is that the literature is not always very clear on this point. Thus, in order for us to decide whether or not intuitions a priori can be identified with internal diagrams, we at least need to know at what level the objects represented by internal diagrams are supposed to live. For otherwise, the question has no "bite" and every answer to it would more or less float in the air.

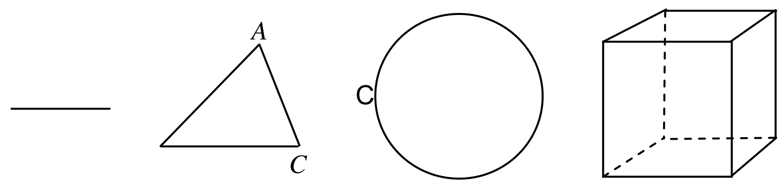
Geometric diagrams. In line with the scope of the present study (§ 1.2), we focus on *geometric diagrams*. Iwasaki makes the following suggestive remarks, indicating some sort of demarcation of geometric diagrams ("geometry diagrams") from other types of diagrams:

> Geometry diagrams have characteristics that are not shared by diagrams in any other domains. Most importantly, geometry diagrams stand for themselves. In other words, they are not abstractions of the real world or anything else that is the real object of interest. In almost every other domain, diagrams represent something other than themselves that one is trying to study (Iwasaki [84], p.661).

 93

Iwasaki's point, we take it, is somewhat as follows: in case of geometric diagrams, we typically take lines, circles, triangles, cubes, etc. to be just that: lines, circles, triangles, cubes, etc..[139] In this respect, geometric diagrams are different from the diagrams as they are typically used in, for example, other branches of mathematics, where we often take diagrams as representing "something other than themselves." For example, in case of diagrams as they are used in category theory, we take an arrow as a morphism between two objects. In elementary set theory, we take, say, an oval (or circle) as a set. In discrete mathematics (e.g., graph theory), we take a line as an edge in a graph, etc. Much more examples could be added.

Below are four examples of geometric diagrams:



(I) (II) (III) (IV)

We note that diagrams (I) and (IV) differ from diagrams (II) and (III) in that in the latter several textual labels occur. In particular, diagram (II) involves the labels *A*, *B*, and *C*; diagram (III) involves the label C.

Hammer [62], pp.11, 13-6, has argued that diagrams typically occur in the context of larger documents that also involves text, and hence language. In this respect, labels would serve to assist cross-referencing between text and diagram (*ibid.*, p.16). Note that this suggests that Hammer mainly considers external diagrams. As the diagrams we are interested in, figure in certain cognitive procedures (i.e., proof procedures) carried out by a mathematician, we do not think that they deserve explicit study. We need not to assume that this mathematician adds these labels himself. In particular, since in Kant's view on mathematical proof language does not seem to play a very significant role, the function of labels that Hammer indicates hardly comes into play. Nevertheless, in what follows, *we* will frequently talk and reason about certain diagrams. To this end, the addition of labels by *us* is extremely convenient. We take it that labels form a means to access diagrams for us talking and reasoning *about* diagrams, and not so much for a mathematician who in Kant's view reasons *with* them. We

[139] Iwasaki furthermore states that it is precisely this property of geometric diagrams making them the ideal object of study (*ibid.* [84], p.660).

may imagine that this mathematician accesses his diagrams by other means, for example, by *attending* to them. Let us address a further issue.

The question may arise whether diagram (IV) is a two-dimensional diagram representing a three-dimensional object (namely, a cube), or that diagram (IV) is itself three-dimensional. The literature appears to be divided on the issue. For example, some are inclined to think of diagrams (insofar as they are spatial representations) as two-dimensional. Such authors have a tendency to think of a diagram in terms of an inscription, or a two-dimensional pixel-like representation (e.g., Furnas [51]). Others, in contrast, also allow that diagrams can be either two- or three-dimensional (cf. above). In this respect, then, diagram (IV) is seen as a three-dimensional diagram. See, for example, Horn [80], pp.28, 67-68. Engelhardt, too, holds that the standard drawing of a cube (i.e., diagram (IV) above) is three-dimensional. Specifically, Engelhardt identifies a diagram with what we *see* in the representation, which, in this case, is a three-dimensional cube (Engelhardt [38], p.21; see also below). We found no indication in the literature indicating that the dimensionality of diagrams may be other than two or three.

In what follows, we tend to go with the latter view. Accordingly, we think of diagram (IV) has having a three-dimensional spatial structure. The others, in contrast, we think of as having a two-dimensional spatial structure. (Diagram (I) we think of as a line in two-dimensional space.) In short, then, diagrams are either of dimension two or three.

We distinguish between a diagram and the medium in which it is realized (cf. above). Diagrams can be realized in different media. For example, diagrams (I)-(IV) above come in the drawn (or printed) medium. As such, they are external diagrams. Alternatively, a diagram may be merely imagined, making it an internal diagram.[140] One may propose that a diagram can also be stored or represented in a computer. Without denying this, it nevertheless seems that the typical spatial properties of a diagram will get lost accordingly (at least on some level of representation). Related to this is the following.

It has been pointed out in the literature that a diagram is expressible as a set of sentences (e.g., Lindsay [105], p.116). Consequently, it may seem that a diagram can also come in the linguistic medium. However, even if we grant this point—though it is not clear whether and why we should—, it may again be said that the typical spatial properties of diagrams would be lost in this case. Perhaps it is more accurate to say that we can *describe* a diagram in terms of a set of sentences. The point is that certain typical diagrammatic features may not be adequately reflected in such a description.

The point can perhaps be rephrased as follows. Although it may be that a diagram and a set of sentences have the same content, they may express it in different modes. A set of sentences expresses a certain content in a linguistic mode (let us call it), while a diagram expresses a content in a diagrammatic

[140] See also A713/B741.

mode. Shimojima has suggested that these differences in mode of expression turn on a match or mismatch between certain structural constraints on a representation (a sentence or a diagram) on the one hand and their targets on the other. These structural constraints on a representation are intimately related on the geometric structure of the medium in which they are realized. See Shimojima [144] for details.

In the light of the above considerations, it seems reasonable to conclude that a diagram is to a considerable extent constituted by features of certain specific media in which it is possibly realized, and not by those of other media (cf. Sloman [148], p.217). These features turn on the *spatial structure* of the realizing medium. If a medium is not in the possession of an appropriate spatial structure, then it seems that a diagram cannot be realized in that medium *as a diagram*. In this case, the structure of the realizing medium is such that it cannot adequately reflect certain specifically *diagrammatic* features of a diagram. Thus, one reason why diagrams cannot be adequately realized in language is perhaps that sentences are typically not in the possession of spatial structure.

Since spatial structure is a geometrical notion, diagrams are in part constituted by the geometry of the realizing medium. The geometric structure of the media in which diagrams are realized is what we call *diagrammatic space*.[141] One can conceive of diagrammatic space in terms of Kant's a priori intuition of space (cf. A22-30/B37-45). Given our terminology, it is certainly in line with Kant's thought to say that the structure of diagrammatic space is Euclidean (A25/B40).

Intuitions as diagrammatic items of knowledge: a reconstruction. We define a diagram as a complex object comprised by a collection of diagrammatic relations among diagrammatic objects in a diagrammatic space (cf. also Engelhardt [38]). We represent diagrammatic space either as $\mathbb{R}^2$ or $\mathbb{R}^3$. Thus, it is not to be assumed that diagrammatic space is $\mathbb{R}^2$ or $\mathbb{R}^3$; in contrast, $\mathbb{R}^2$ or $\mathbb{R}^3$ are two models of diagrammatic space. In the former case, we refer to diagrammatic space as *two-dimensional space*; in the latter case, we refer to it as *three-dimensional space*. We assume that the geometrical structure of these spaces is Euclidean, something which is certainly not uncongenial to Kant's views on space (cf. A25/B40-1).

Since, because of certain contextual factors, the same configuration of diagrammatic relations among diagrammatic objects can determine different diagrams (cf. above), we also take these contextual factors as constitutive for a diagram. Somewhat more precisely, then, we can represent a diagram as a quadruple $(\Sigma, \mathfrak{R}, C, \gamma)$. Here, Σ is a diagrammatic space, $\mathfrak{R}$ is a set of diagrammatic relations in Σ among the diagrammatic objects in C, and γ is a

[141] This terminology is adapted from Engelhardt [38], pp.21-2, who understands it somewhat differently.

context (see below). A diagram *is* not a quadruple $(\Sigma, \mathfrak{R}, C, \gamma)$. In contrast, such a quadruple records certain relevant features of a diagram.

Occasionally, we wish to add labels to diagrams in order to facilitate talking and reasoning about diagrams. We wish to think of labels as a specific type of diagrammatic objects. We can frame the following definition. A *labeled diagram* is a quadruple $(\Sigma, \mathfrak{R}, C \cup \Lambda, \gamma)$. Here, Λ is set of diagrammatic objects whose members we refer to as *labels*. The quadruple $(\Sigma, \mathfrak{R}, C, \gamma)$ is a diagram in the usual sense (cf. above). As suggested earlier, it should be understood that labels are typically added to diagrams by *us* talking and reasoning about diagrams, and not necessarily by a mathematician who, in Kant's view, reasons with them. Let us provide some further elucidation.

Consider any labeled diagram. Examples of diagrammatic objects are those we refer to as *points*, *lines*, *curves*, or *labels*. Labels typically are textual objects such as letters (e.g., *A*, *B*, *C*, etc.). Diagrammatic relations are spatial relations—or so we have assumed. By this we do not mean relational *expressions* such as, for example, *x lies to the left of y* or *u lies above v*.[142] In contrast, diagrammatic relations are relations *in space*—diagrammatic space, that is. Thus, the following may be considered as a representation of two respective examples of diagrammatic relations (in two-dimensional space):

u

$x \qquad\qquad y$

v

The leftmost diagrammatic relation can be *described* in terms of the relational expression *x lies to the left of y*; the rightmost diagrammatic relation can be *described* in terms of the relational expression *u lies above v*. Diagrammatic objects are objects entered on the places of the variables.

Engelhardt [38], § 2.5, discusses a broad inventory of diagrammatic relations. We consider only the following: *spatial clustering* (of which *labeling* forms a specific case), *lineup*, and *linking*.

1. *Spatial clustering* is the spatial grouping of several different diagrammatic objects into a separate cluster by way of a proximity relation. Diagrammatic objects that are related according to such a proximity relation are said to form a *spatial cluster*.

[142] Adding variables where objects (relata) can be entered.

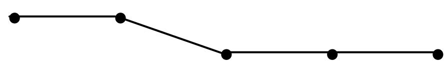
97

Labeling is specific case of spatial clustering. As an example, reconsider diagram (III), a geometric diagram we presented earlier. In this case, the letter C forms a spatial cluster together with a closed curve. Naturally enough, we say that the letter C labels the circle represented.

A caveat must be mentioned, however. In order let C be the label and the circle be the labeled object (and not, for example, the other way around) the object C needs to have a different so-called *syntactic role* than the circle. See Engelhardt [38], 32, and 74-8 for further discussion.

2. In case of a *lineup*, several diagrammatic objects are related in terms of a string. A lineup may be ordered or unordered. In case of an *ordered lineup*, a permutation of the diagrammatic objects constituting the lineup changes the diagram; in case of an *unordered lineup* the diagram will not change. An example of an ordered lineup is a comic strip. An example of an unordered lineup is a sequence of icons on a window of a word processor.

3. *Linking* involves several diagrammatic objects any of which functions as a *node* or as a *connector*. (These correspond to two respective syntactic roles. See above.) For example, lines or arrows can link several points. When such a configuration involves no branching it is called a *linear chain*. An example of a linear chain is the diagram below:



When a configuration of nodes and connectors forms a closed loop, it is called a *closed chain*. A *tree* is a linking configuration that involves branching from one node (the root) and involves no circular chains. A *network* is a linking configuration that involves at least one closed loop. It follows from this definition that every closed chain is a network. An example of a closed chain is a diagram representing a triangle. We may say that the nodes represent the corner points and the connectors represent the sides. Examples of networks are diagrams (IV), (V) and (VI) above.

What a diagram is depends in part on the intentions of a possible user of a diagram. In order to illustrate this, reconsider diagram (II) above. It is natural to refer to this diagram as a triangle (or triangle *ABC*), especially within the current context (i.e., a discussion of geometric diagrams). However, within the context of a discussion on, for example, graph theory, we may equally naturally refer to it as a complete planar graph with three nodes labeled *A*, *B*, and *C*. It seems that the context in which it is produced and used in part constitutes that diagram. We think that this context at least includes the intentions of the user of a diagram.

Accordingly, the notion of diagram we have described is in some sense non-reductionistic. In particular, a diagram is not anything neutral like an inscription or a collection of inscriptions. In contrast, a diagram corresponds to what we "see" in an inscription. What we see in an inscription involves context. We do think that Kant's notion of intuition also involves the kind of intentionality we have in mind here (cf. § 3.2). What an intuition is depends at least on the intentions of an agent. Changing these intentions changes the intuition. See also § 4.2, where we speculate on the workings of intentionality.

Suppose a diagram δ is given. Which objects are recognized as diagrammatic objects constituting δ, will generally depend on certain contextual factors. For example, the diagrammatic objects recognized in diagram (II) above may be three lines and the labels A, B, and C. Alternatively, the diagrammatic objects recognized may be a triangle and the three labels. In the latter case, we consider the triangle as primitive, in the former case as composite.

There are still other interesting possibilities. Thus, in case of diagram (II), one may also recognize a certain region of two-dimensional space as a diagrammatic object, namely, the region enclosed by the lines AB, BC, and AC. Something along these lines seems to be suggested by Euclid's definition of a triangle. Thus, Euclid defines a *line* as a "breadthless length." A *straight line* "is a line which lies evenly with the points on itself." A *boundary* is "an extremity of anything." A figure is defined as that what is contained by any boundary or boundaries. Finally, then, triangle is a figure contained by three straight lines (Euclid [41], I, pp.153-4). The impression we get is that Euclid conceives of a triangle more or less as a certain region of the plane.

Consider also consider Hilbert's definition of a triangle. Hilbert [69], p.5, defines a *line*[143] as a "system of two points."[144] A line is referred to as, for example, AB (or, what Hilbert takes to be the same line, BA). A and B are called the *endpoints* of the line. A *line combination*[145] is a "system of lines" AB, BC, CD,…, KL, connecting endpoints A and L of respectively the lines AB and KL. A *polygon* is a line combination AB, BC, CD,…, KL such that A and L coincide. The lines AB, BC, etc are called the *sides* of the polygon. Finally, a polygon with three sides is called a *triangle*. (Cf. *ibid*., p.9)

It appears that Hilbert conceives of a line as something like a set of two points (the end points of the line). In the light of this, a triangle is conceived of as something like a set of three lines, that is, as a set any member of which is a set consisting of two points. In diagrammatic terms, then, Hilbert would perhaps recognize the points A, B, and C as diagrammatic objects, or perhaps something like the sets $\{A, B\}$, $\{B, C\}$, and $\{A, C\}$.

[143] Hilbert's own German term is *Strecke*.
[144] The term *point* remains undefined; cf. Hilbert [69], p.2.
[145] Hilbert's own German term is *Streckenzug*.

What we observe is that Euclid and Hilbert appear to have two different *conceptions* as to what a triangle is (these conceptions manifest themselves in the respective definitions they provide). In the light of this, the point we wish to make is that, in the case we are currently considering (i.e., diagram (II) above), what diagrammatic objects are recognized will in part depend on one's conception of a triangle. Let us address a final issue.

Consider again diagram (II) above. In order to fix our thoughts, suppose we adopt the first option. Then we may refer to the lines as AB, BC, and AC respectively. These three lines and the three labels are interrelated by way of several diagrammatic relations we refer to as $R_1,\ldots, R_n$.[146] Collectively, given a context γ (about which we will not bother now), may represent diagram (II) as follows:

$$\tau = (\mathbb{R}^2, \{R_1,\ldots, R_n\}, \{AB, BC, AC\} \cup \{A, B, C\}, \gamma).$$

Besides the three lines AB, BC and AC (and the labels A, B, and C), it is natural to distinguish other diagrammatic objects. For example, the three corner points of the triangle may also be recognized as diagrammatic objects. These corner points may be referred to as A, B, and C respectively. Note that we now do not think of the labels primarily as constituents of the diagram τ. In contrast, we use the labels as a means to refer to the aforementioned corner points.

The corner points A, B and C, however, are strictly speaking not diagrammatic objects in τ. We call such objects *emergent diagrammatic* objects. Roughly, emergent diagrammatic objects are diagrammatic objects that can be recognized provided that others diagrammatic objects are. Besides the corner points A, B, and C, other emergent diagrammatic objects may be recognized, for example, $\angle ABC$, $\angle BAC$, and $\angle ACB$. Also, triangle ABC may be recognized as an emergent diagrammatic object in this case.

Once we have emergent diagrammatic objects, we also have the possibility of what we may call *emergent diagrammatic relations*. For example, once we have recognized $\angle ABC$ and $\angle BAC$ as (emergent) diagrammatic objects, one may also wish to speak in terms of a relation of *incidence* between those two angles.

Dynamic diagrams. Diagrams, as we've seen them thus far, are static objects. However, there is also the possibility of diagrams that change through time. Such diagrams may be called *dynamic*.[147] The relevance of considering diagrams lies in the fact that, in Kant's view, intuitions are constituted by a configuration of relations in space and *time*.[148] In order to facilitate our discussion, we assume

[146] Assuming that there are a finite number of such relations.
[147] Nowadays, dynamic (or interactive) diagrams have been studied mainly in the area of computer graphics. See Card, Mackinlay, Schneidermann [25].
[148] See especially A162-3/B203 for an illustration for the temporal nature of intuitions.

discrete time. That is, we shall assume that time consists of a succession of discrete instants.

More elaborately, a dynamic diagram is a diagram that changes through time by means of the successive addition or deletion of diagrammatic objects and diagrammatic relations by an agent. Thus, a dynamic diagram is one that evolves. We can understand the dynamics of diagrams in terms of several operations that are performed on it. Given a diagram, an agent operates on it and thus produces another diagram.

At any instant of time, the development of a dynamic diagram is in a certain stage. This stage can be taken as a static diagram. Accordingly, we can trace a dynamic diagram by recording the successive stages enters in the course of its development. We often tend to refer to such a stage as a *diagram stage*. A diagram stage can be seen as a dynamic diagram projected on a time instants.

In the present study, dynamic diagrams are extremely important. In regard of Kant's views on proof, the dynamics of the diagrams we will meet comes precisely as the result executing several constructive procedures. These constructive procedures either create a diagrammatic representation or modify a once created diagrammatic by performing a certain operation upon it. Thus, for example, in the course of a typical proof in geometry, one starts by creating a diagram of a triangle (say) and subsequently operates upon it by adding several auxiliary lines (see chapter 4, especially § 4.2).

Concluding remarks. In view of § 3.3 (where we discussed Kant's notion of construction), the main conclusion we draw from this section is that the product of a constructive process is a non-propositional, diagrammatic item of knowledge, i.e., an intuition. To construct a concept, therefore, means to exhibit that concept diagrammatically, in terms of an intuition a priori. We have construed an intuition as a configuration of spatial relations among diagrammatic objects in a diagrammatic space. The diagrammatic relations naturally correspond to what Kant calls the form of an intuition. The diagrammatic objects, in turn, naturally correspond to their content. Diagrammatic space, furthermore, more or less corresponds to Kant's a priori intuition of space. In the next chapter (especially § 4.2), we shall see how, in Kant's view, the diagrammatic structure of intuitions a priori is *employed* in the course of proving a theorem.

§ 3.5. Conclusions

In the present chapter, we investigated two crucial elements of Kant's philosophy of mathematics: Kant's notion of intuition and the intimately related notion of construction. In Kant's view, a mathematical proof essentially proceeds constructively. This means that a mathematician, *qua* mathematician, essentially proves his theorems by means of constructing concepts. To construct a concept

101

means to exhibit a priori the intuition to that concept. The intuition produced by a process of exhibition a priori is itself an intuition a priori.

In Kant's view, an intuition is an item of knowledge that is singular and immediate. Kant distinguished at least three different kinds of intuitions: intuitions a posteriori, intuitions a priori, and pure intuitions. Intuitions a priori play a crucial—indeed, essential—role in mathematics. Within the context of mathematics, intuitions a priori coincide with pure intuitions, that is, any intuition a priori playing a role in mathematics is a pure intuition and *vice versa*. Empirical intuitions come into play in what may be broadly referred to as the empirical sciences.

As to intuitions generally, we can distinguish between form and content. The former turns on the spatio-temporal organization of the latter. The form of an intuition resides in the knowing subject. In case of an intuition a posteriori, the content arises from experience. In case of an intuition a priori in mathematics, the content arises is produced by the productive imagination. It is precisely this point that makes it hard to compare an intuition a priori in mathematics with a state or episode of visual perception, since the latter typically involves the processing of sensations that come as the result of environmental actions on an agent's sensory apparatus. The important role of the productive imagination in mathematical proof forms an aspect of Kant's views on mathematical proof that has been overlooked by many Kant commentators.

The intuitions that form the product of constructive processes are fundamentally characterized by their being in space and time, which suggests that intuitions are non-propositional, diagrammatic items of knowledge. In the light of this, we have construed an intuition as a configuration of diagrammatic relations among diagrammatic objects in a diagrammatic space. Intuitions are made dynamic by adding a temporal dimension.

Chapter 4

Proof in action

The aim of the present chapter is to provide a thorough analysis (or interpretation) of The Passage. We shall carry out our analysis in the light of a methodological framework for proving theorems in mathematics due to the neo-Platonic philosopher Proclus (410-485 A.D.[149]). Accordingly, this chapter consists of two main sections. We present and discuss Proclus' methodological framework in § 4.1. Next, in § 4.2, The Passage is analyzed in terms of this framework. A concluding section brings this chapter to and end (§ 4.3).

§ 4.1. Proclus' methodological framework

This section consists of three parts. In the first part, we shall present and discuss Proclus' framework. In the second part, we will provide an elucidation. In the third concluding part we open up a few broader issues.

Presentation and discussion. Proclus discusses his framework in his famous commentary on the first book of Euclid's *Elements* [129]. It is summarized in the following:

> Every problem and every theorem that is furnished with all its parts should contain in itself all of the following: an enunciation, an exposition, a specification, a construction, a proof, and a conclusion (Proclus [129], p.159).

Proclus sees every problem and every theorem ideally ("furnished with all its parts") as broken up into six "parts" (see the next section for an elucidation). We make three terminological points.

1. As can be seen from the citation above, one of the parts of a "theorem or problem" is called the *proof*. This word is a translation of the originally Greek word *apodeixis* (see Euclid [41], I, p.129). Motivated by this, we shall henceforth

[149] Cf. Heath, in Euclid [41], I, p.29. See also Morrow, in Proclus [129], xvi n.6.

use the term *apodeixis* rather than *proof* in order to refer to the relevant part of a problem or theorem.

The reason is that, from a modern point of view, it is more appropriate to refer to a proof as that what is comprised by what Proclus calls the exposition, specification, construction, *apodeixis* (using our terminology) and conclusion (see below). Accordingly, the term *proof* would suffer from overload had we used it also in order to refer to what we have now called the *apodeixis*.

2. What Proclus calls a *problem* we henceforth call an *existence theorem*.[150] This terminological switch gives the impression that existence theorems are simply treated as a special kind of theorems. This is indeed what we intend (see point also point b. below). Accordingly, we won't have to keep track of theorems as well as problems but we simply treat them as one of a kind. This will greatly facilitate our discussion. Henceforth, when we speak of theorems, we mean to include existence theorems as well.

The historical question to what extent this terminological point forms an adequate representation of Proclus' own views (or anyone else's views) is something that shall not be treated here.[151]

3. In the citation above, one of the parts of a theorem or existence theorem is called the *construction*. Here, construction is traditionally often understood as mechanical construction, for example, by means of ruler and compass. It turns out that construction in this sense is typically different from the way Kant understands the term *construction* (see § 3.3). It turns out, however, that Kantian construction, within the context of traditional geometry, can be naturally associated with what Proclus calls construction (§ 4.2). This motivates us not too choose a different term for what Proclus calls construction in order to distinguish it from Kantian construction. As a consequence, the term *construction* now becomes somewhat ambiguous. We do not consider this very harmful, however. The context will always make clear what sense of *construction* we have in mind. For more on Proclus' understanding of the term *construction*, see the elucidation below.

Collectively, then, we find that every theorem (existence theorems included) ideally consists of the following six parts (henceforth referred to as *items*):

---

[150] The term *construction problem* is also used in the literature.

[151] For example, Proclus himself clearly suggests that there is a difference between existence theorems (i.e., what he calls problems) and theorems when he says:

> Again the propositions that follow from the first principles he [i.e., Euclid] divides into problems and theorems, the former including the construction of figures, the division of them into sections, subtraction from and additions to them, and in general the characters that result from such procedures, and the latter concerned with demonstrating inherent properties belonging to each figure (Proclus [129], p.63).

- enunciation,
- exposition,[152]
- specification,
- construction,
- *apodeixis*,
- conclusion.

For convenience, let us refer to the above list as *Proclus' methodological framework for proving theorems in mathematics*, or *Proclus' framework* for short. That Proclus' framework is indeed a methodological framework for proving theorems in mathematics will be argued below.

We need to discuss a few more terminological points.

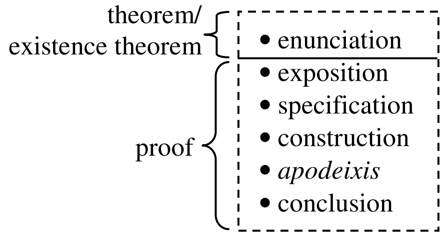
a. Looking at, for example, a mathematics textbook or journal article, a proof is typically presented in combination with a statement of the theorem proved.[153] It will turn out that the statement of a theorem can be associated with what Proclus calls the enunciation. Furthermore, the proof can be naturally associated with the remaining five parts of Proclus framework—or so we shall argue. Therefore, the modern subdivision in terms of theorem and proof can be easily seen as a kind of superstructure imposed on Proclus' own six-fold subdivision. Both Proclus' framework and a copy of it with the imposed superstructure are depicted below:

theorem/
existence theorem

- enunciation
- exposition
- specification
- construction
- *apodeixis*
- conclusion

Proclus' framework

[152] Instead of exposition, the term *ecthesis* is also sometimes used in the literature. E.g., Hintikka and Remes [79].
[153] See the appendix for examples.



Proclus' framework from a
modern point of view

In what follows, we shall use the modern point of view. Hence, we think of a proof as (ideally) consisting of five items.

b. Earlier, we introduced the term *existence theorem*. How it is that we can treat existence theorems as a kind of theorems? From a more traditional point of view, a construction problem can be seen as a request to produce a certain object, typically by some mechanical means such as a ruler and a compass. For example, a well-known existence theorem asks us to produce an equilateral triangle on a given finite straight line by means of a ruler and a compass (cf. Euclid [41], I, p.241).[154] Accordingly, an existence theorem is not primarily something—i.e., a theorem—that one proves to be true.

However, from a more modern point of view, an existence theorem can be easily reformulated in such a way that it becomes a genuine theorem. For example, the existence theorem mentioned above can be reformulated as follows: "For every straight-line segment, there exists an equilateral triangle having that line segment as one side." One now easily sees why the term *existence theorem* is used. What is required is to prove that a certain object exists.

One way of proving this theorem is by producing an equilateral triangle as required by means of a ruler and a compass (cf. Euclid [41], I, pp.241-2). However, there may be alternative ways to prove this theorem. For example, one may suggest proving it by means of a *reductio ad absurdum*, in broad outlines as follows. One first assumes that the required equilateral triangle does not exist and subsequently derives a contradiction from that assumption. Hence, it exists.

If one finds this an acceptable strategy (which not everyone does), then one may hold that, according to this strategy, the existence of the equilateral is being proved without producing one. At any rate, how one specifically proves an

[154] In this case, the ruler is an unmarked straight edge.

existence theorem is not so important for us, however. Again, what is important is that one can treat the original construction problem as a theorem to prove.

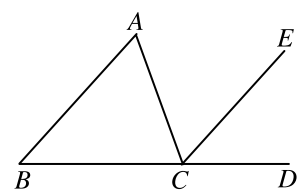
Elucidation. In this section we elucidate the respective items of Proclus' framework. We proceed as follows. Given any item, we first consider what Proclus himself says on the point; subsequently we provide an illustration by referring to the corresponding part of a theorem or proof from Euclid's *Elements*. In this respect, we will take theorem 32 from the first book of the *Elements* (henceforth simply referred to as *theorem 32*) and its proof as our example.

Euclid's theorem 32 is in effect a identical to theorem 1 from § 1.1. Also, Euclid's proof matches Kant's description of the proof in The Passage closely. We shall exploit this in our analysis of The Passage in § 4.2.

This is Euclid's statement of theorem 32:

> *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles* (Euclid [41], I, p.316).

He subsequently presents the following proof (*ibid*., pp.316-7):

> Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*;
>
> I say that the exterior angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*, and the three interior angles of the triangle *ABC*, *BCA*, *CAB* are equal to two right angles.
>
> For let *CE* be drawn through the point *C* parallel to the straight line *AB*. [I. 31]
>
> 
>
>
> Then, since *AB* is parallel to *CE*, and *AC* has fallen upon them, the alternate angles *BAC*, *ACE* are equal to one another. [I. 29]
>
> Again, since *AB* is parallel to *CE*, and the straight line *BD* has fallen upon them, the exterior angle *ECD* is equal to the interior and opposite angle *ABC*. [I. 29]
>
> But the angle *ACE* was also proved equal to the angle *BAC*; therefore the whole angle *ACD* is equal to the two interior and opposite angles *BAC*, *ABC*.
>
> Let the angle *ACB* be added to each; therefore the angles *ACD*, *ACB* are equal to the three angles *ABC*, *BCA*, *CAB*.
>
> But the angles *ACD*, *ACB* are equal to two right angles; [I. 13]
> therefore the angles *ABC*, *BCA*, *CAB* are also equal to two right angles.
>
> Therefore etc.
>
> Q.E.D.

The expressions between square brackets are taken over from Heath's edition of the *Elements*. They indicate Euclid's appeal to other theorems (or postulates, or common notions, or definitions). For example, the expression '[I. 31]' means that

Euclid has made an appeal to theorem 31 from the first book of the *Elements*. In case of other proofs (but not in case of the example we are currently considering), we also find expressions such as '[*C.N.* 1]' or '[Post. 5],' indicating an appeal to a common notion or a postulate respectively. Also, expressions referring to a definition are found; these are indicated by, for example, '[Def. 10]' (e.g., Euclid [41], I, p.269). Let us now clarify the status of Proclus' framework.

Very likely, Proclus formulated his framework based on a written version of (the first book of) the *Elements*. Accordingly, the impression may arise that Proclus' framework primarily aims at describing a certain type of texts, namely, the proofs as they are written in the *Elements*. We think that this way of looking at Proclus' framework is somewhat one-sided. For as it will turn out, Proclus' explanations of the respective parts of his framework suggest a somewhat different perspective. Specifically, Proclus strongly suggests that the items of his framework concern certain types of procedures that are carried out when a Euclidean theorem is proved (e.g., the *making* of a construction, or the *making* of an inference, etc.; see below for details). The impression arises that Proclus' framework is a *methodological framework* for proving Euclidean theorems. We will return to this point at the end of our elucidations.

Let us now explain the successive items of Proclus' framework, beginning with the enunciation. We make a few observations along the way, without taking a definite stand on the issues raised. Most of these issues will receive due attention in our analysis of The Passage (see below).

1. The enunciation. As to the enunciation, Proclus says:

> […] the enunciation states what is given and what is being sought from it, for a perfect enunciation consists of both these parts (Proclus [129], p.159).

Apparently, an enunciation is twofold: it concerns something that is supposed to be given as well as something that is supposed to be sought from it. Put differently, we may say that an enunciation concerns something given on the one hand and a goal to be achieved on the other. A first impression that may accordingly arise is that the enunciation sets forth a certain *task* or *problem*: a goal that needs to be reached from something given. This point will be elaborated on in § 5.4.

In view of the example we are currently considering, it is natural to associate the enunciation with Euclid's statement of theorem 32.

Setting details aside, can we distinguish the two aforementioned components in case of Euclid's statement of theorem 32. In other words, considering the enunciation of theorem 32, can we distinguish between what is given and what is sought? Broadly speaking, what is given appears to be something like "any triangle" (or "a triangle" or "triangles"[155]), or any triangle having one of its sides extended.

In case of theorem 32, the goal ("what is sought") seems twofold. In particular, the goal can be associated with two propositions, namely

(i) that the external angle is equal to the two opposite and internal angles
(ii) that the sum of the internal angles is equal to two right angles.

Apparently, then, to reach the goal (or "to find what is sought") means: to establish (i) and (ii).

2. The exposition. As regards the exposition, Proclus says:

> The exposition takes separately what is given and prepares it in advance for use in the investigation (*ibid.*).

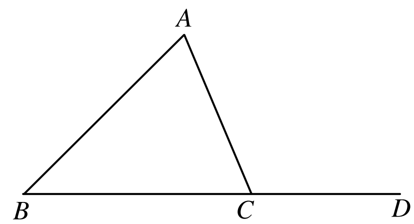
In the exposition, one takes apart one component of the enunciation, namely, that what is supposed to be given. Furthermore, Proclus suggests, a certain preparation takes place of the component that is taken apart. Furthermore, the "investigation" Proclus refers to presumably concerns the items 2-4 of Proclus' framework.

In view of what Euclid has written down for us in the *Elements*, two things seem relevant to consider in this respect, namely a sentence fragment and a diagram. First, we have Euclid saying:

> Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*; […].

Second, and related to this, we have the following diagram:

[155] Heath's translation of the *Elements* speaks in terms of *any triangle* and accordingly uses the indefinite pronoun *any*. However, he doesn't so generally on other occasions. Therefore, no general conclusions can be based on Euclid's use of the indefinite pronoun on this particular occasion. For example, Euclid sometimes alternatively uses the indefinite article *a*. See, e.g., his statement of theorem 6 from the first book of the Elements: "*If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another*" (Euclid [41], I, p.255). At other occasions, he uses neither *any* nor *a* but simply speaks of "triangles" and the like. For example, theorem 5 of the same book is formulated as follows: "*In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another* (Euclid [41], I, p.251). As far as we can see, Euclid does not appear to use quantifiers such as *all* or *every* in this respect.



We note that the extension of *BC* to the point called *D* is secured by *Euclid's second postulate* which says that every finite straight line can be continuously extended in a straight line (Euclid [41], I, p.154). We do not find an indication of this in the quotation above. Furthermore, one may speculate that Euclid's "Let *ABC* be a triangle" together with the corresponding constituent of the above diagram in one way or another involves an appeal to Euclid's definition 19, which says, among other things, that a triangle is a figure enclosed by three straight lines (cf. Euclid [41], I, p.154).[156] This is neither indicated in the quotation.

3. The specification. Proclus says:

> The specification takes separately the thing that is sought and makes clear precisely what it is (Proclus [129], p.159).

In other words, the specification takes apart and makes precisely clear the other component of the enunciation, namely, the goal that has to be reached. Note that, in Proclus' view, the specification apparently involves an element of clarification: the specification not merely states the goal—it *makes clear precisely* what it is. We suspect that this element of clarification takes place in view of the diagram presented above: once the diagram is given, it can be made precisely clear what needs to be proved. In this respect, the enunciation on itself is considerably less clear.

In case of Euclid's proof of theorem 32, the specification seems to concern the following:

> I say that the exterior angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*, and the three interior angles of the triangle *ABC*, *BCA*, *CAB* are equal to two right angles.

---

[156] Written out in full, Euclid's definition 19 reads: "*Rectilineal figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines" (*ibid*.).

The impression may arise that the specification refers to the diagram that appeared to be associated with the exposition. (These references are facilitated by means of several combinations of labels.) In this respect, note in particular that Euclid refers to several spatial relations that constitute the diagram. For example, one angle is referred to as *exterior*, others as *interior*, or *interior and opposite*.[157] The reference to these spatial relations would seem to make no sense apart from a diagram in which these spatial relations occur as constituents.

4. The construction. Proclus says:

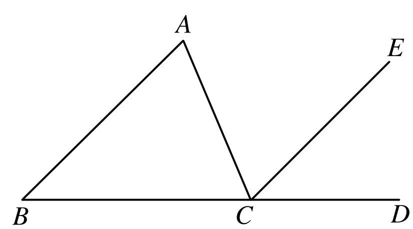
> The construction adds what is lacking in the given for finding what is sought (*ibid*.).

Apparently, the construction modifies that what is given in the exposition.

In the case of the proof of theorem 32, two things seem relevant for the construction. First, we have Euclid saying:

> […] let *CE* be drawn through the point *C* parallel to the straight line *AB*. [I. 31]

Second, and related to this, we have the addition of a line to the diagram associated with the exposition. This results in a new diagram, for example, the diagram below:



The role of the diagram in particular is not obvious. The further remarks that can be made are similar to those made on behalf of the exposition.

We note that the construction involves an appeal to theorem 31 from the first book of the *Elements* (as indicated by the text). Let us quote Euclid's own "enunciation" of this theorem:

> *Through a given point to draw a straight line parallel to a given straight line* (Euclid [41], I, p.315).

---

[157] See § 3.4.

Given the terminology we have chosen in this study, this may be taken as an example of an existence theorem. In the light of this, it can be reformulated thus:

THEOREM 31. Given a point P and a straight line ℓ, there exists a line through P parallel to ℓ.

In the history of Euclidean geometry, this theorem has been taken as a substitute for Euclid's axiom of parallels. When taken accordingly, theorem 31 is more commonly known as *Playfair's axiom*.

***5. The* apodeixis.** Proclus says:

> The proof [i.e., *apodeixis*] draws the proposed inference by reasoning scientifically from the propositions that have been admitted (Proclus [129], p.159).

In drawing the inference, a mathematician typically uses other theorems that have already been proved ("admitted").

Proclus seems to suggests that only one inference is being drawn in the *apodeixis*. However, one naturally feels inclined to say that, Euclid draws several inferences:

> Then, since *AB* is parallel to *CE*, and *AC* has fallen upon them, the alternate angles *BAC*, *ACE* are equal to one another. [I. 29]
>
> Again, since *AB* is parallel to *CE*, and the straight line *BD* has fallen upon them, the exterior angle *ECD* is equal to the interior and opposite angle *ABC*. [I. 29]
>
> But the angle *ACE* was also proved equal to the angle *BAC*; therefore the whole angle *ACD* is equal to the two interior and opposite angles *BAC*, *ABC*.
>
> Let the angle *ACB* be added to each; therefore the angles *ACD*, *ACB* are equal to the three angles *ABC*, *BCA*, *CAB*.
>
> But the angles *ACD*, *ACB* are equal to two right angles; [I. 13]
> therefore the angles *ABC*, *BCA*, *CAB* are also equal to two right angles.

To be exact, five inferences can be distinguished. Remaining close to the original text, we can formulate these respective inferences as follows:

(1) Line *AB* is parallel to *CE*, and *AC* intersects both. Therefore,

$$\angle BAC = \angle ACE.$$

(2) Line *AB* is parallel to *CE*, and line *BD* intersects both. Therefore

$$\angle ECD = \angle ABC.$$

(3) $\angle ACE = \angle BAC$. Therefore,

$$\angle ACD = \angle BAC + \angle ABC.$$

(4) $\angle ACD = \angle BAC + \angle ABC$. Therefore,

$$\angle ACD + \angle ACB = \angle BAC + \angle ABC + \angle ACB.$$

(5) $\angle ACD + \angle ACB = 180°$. Therefore,

$$\angle BAC + \angle ABC + \angle ACB = 180°.$$

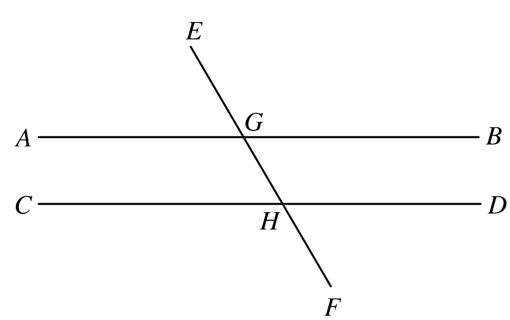
We note that, as an addition to inference (1), we find a reference to theorem 29 from the first book of the *Elements*:

> *A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles* (Euclid [41], I, p.311).

The theorem may be may be more conveniently formulated thus:

THEOREM 29. If a line EF intersects two parallel lines AB and CD, in points G and H respectively, then (α) $\angle AGH = \angle GHD$, (β) $\angle EGB = \angle GHD$, and (γ) $\angle BGH + \angle GHD = 180°$.

The diagram below provides clarification:



Inference (2) is annotated by a reference to the same theorem.

There is no similar addition to inference (3), but we may suspect that it appeals to Euclid's common notion 2:

> If equals be added to equals, the wholes are equal (*ibid*., p.155).

A similar point holds for inference (4).

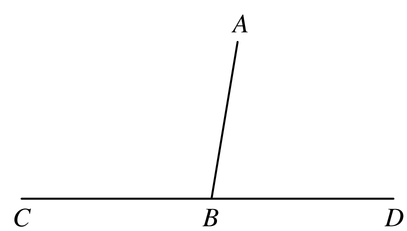
Finally, as an addition to inference (5), we find a reference to theorem 13 from the first book of the *Elements*:

> *If a straight line is set up on a straight line make angles, it will make either two right angles or angles equal to two right angles* (*ibid*., p.275).

The theorem may be more conveniently formulated as:

THEOREM 13. If a line AB is set upon a line CD, then both $\angle CBA$ and $\angle ABD$ are a right angle or $\angle CBA + \angle ABD = 180°$.

The diagram below provides clarification:



6. The conclusion. Finally, we arrive at the conclusion, of which Proclus says:

> The conclusion reverts to the enunciation, confirming what has been proved (Proclus [129], p.159).

Although Proclus is very brief on this point, it appears that in the conclusion a mathematician finally recollects what has been achieved previously. By reconsidering the enunciation, he makes clear that the theorem that had to be proved has indeed been proved.

It seems difficult to illustrate the conclusion by means of Euclid's proof of theorem 32. Perhaps Euclid suppressed the conclusion at the moment he wrote "Therefore etc." It is not obvious, however, what Euclid means by "etc." Perhaps Euclid meant to say something to the effect that every triangle is such that the sum of its internal angles is equal to two right angles. This proposition, then, belongs to what Proclus calls the conclusion. Clearly, however, the *inference* to this proposition does not belong to the conclusion. For, as Proclus himself indicates, inference belongs to the *apodeixis*.

In the light of this, Proclus makes the following interesting remark:

> […] mathematicians are accustomed to draw what is in a way a double conclusion. For when they have shown something to be true of a given figure, they infer that it is true in general, going from the particular to the universal conclusion (*ibid*., p.162).

What Proclus seems to have in mind is that the conclusion is drawn with reference to a specific diagram (figure). Presumably, this is the diagram associated with the exposition, or the construction (see above). Next, a general conclusion is drawn. Perhaps, then, we must say that now the conclusion is drawn with reference to every diagram (or all diagrams) that could result after the exposition and construction.

It remains somewhat unclear why Proclus speaks of "what is in a way a *double* conclusion" (emphasis added). The impression arising is that Proclus means to say that the general conclusion is in a sense nothing but a repetition of the conclusion true of that one diagram. Furthermore, note that Proclus suggests that the conclusion is *about* a (or any) diagram, or every diagram. Indeed, Proclus speaks of showing something to be true *of* a given diagram. Presumably, then, the previous parts where also about that diagram (except, perhaps, for the enunciation and possibly also the exposition).[158]

Concluding remarks. Let us take a few steps backwards and make a few general points.

1. Let us first readdress and confirm a point that was made earlier. In view of what Proclus' has said, we can now clearly see that several parts of his framework comprise a more or less foxed collection of procedures that are carried out in order to prove a theorem. Again, this entitles us to call Proclus' framework a *methodological framework* en henceforth look at it accordingly.
2. The method followed by Euclid in order to prove theorem 32 is considerably clear-cut, especially in the light of Proclus' framework. In particular, it appeared that Euclid treats the items of Proclus' framework more or less successively: Euclid starts with the enunciation; next comes the exposition, the specification, and so on. We must not expect, however, that every statement of a theorem and a proof proceed in such a neat manner. See our analysis of The Passage below.
3. The precise range of Proclus' methodological framework is not clear. Recall that Proclus' presented his framework in a commentary on the first book of Euclid's *Elements*. So, it would seem to apply at least to the theorems and proofs presented there. It would seem reasonable enough, though, to believe that Proclus' framework applies to the whole of the *Elements* and not only to the first

[158] This suggests that, in Proclus' view, the objects of geometry are diagrams (or figures).

book. Consequently, Proclus framework is a methodological framework for proofs at least for the whole of elementary traditional geometry.

Whether Proclus framework is intrinsically tied to elementary traditional geometry and is therefore only relevant in this specific setting is an interesting question to consider. For example, it is interesting to wonder to what extent Proclus' framework applies to proofs in contemporary mathematics. However, we shall not discuss this point here.

4. At least two views on the method followed by Euclid are possible. On the one hand, one can think of the methods followed by Euclid in order to prove theorem 32 as being reflected primarily by the text. For example, one may more specifically think that the text reflects the making of assumptions and several logical inferences. In this respect, on may proceed, the diagram provides merely a convenient but dispensable aid to illustrate this method. For example, commenting on a proof from Euclid's *Elements*, Clark Glymour says:

> "The proof is like a short essay in which one sentence follows another in sequence. […] The proof comes with a picture […]. The picture illustrates the idea of the proof and makes the sentences in the proof easier to understand. Yet the picture itself does not seem to be part of the argument for the proposition [i.e., theorem], only a way of making the argument more easily understood (Glymour [59], pp.14-5).[159]

Although Glymour's words on the role of the diagram are cautionary, he clearly suggests that a proof is like a piece of text ("a short essay"). The diagram forms merely a psychological means in order to facilitate the interpretation of the text.[160] However, one may alternatively propose that the proof primarily concerns the diagram and not the text. Thus conceived, the relation between text and diagram appears to be a rather different one. One may even go further and suggest that the text must be primarily understood as elucidating the proof, which is in itself diagrammatic.

As regards the question "what is the proof," one primarily focuses on the text in the first case. In the second case, one primarily focuses on the diagram. Many logicians and philosophers nowadays seem to take the first rather than the second road. A proof is something that primarily relates to text; diagrams merely form a psychological aid in order to understand a textual proof. Again, in the first case, a proof is something that turns on language (sentences); in the second case, a proof turns on diagrams.

[159] Glymour makes his remark on behalf of a quotation of Euclid's theorem 1 and the corresponding proof (cf. Euclid [41], I, pp.241-2). It is clear, however, that his remark at least applies to the theorems and proofs to be found in the *Elements* generally and that Glymour intends it to do so.

[160] Mueller [112], in contrast, holds that diagrams play an essential role in the proofs from the *Elements*.

Glymour's almost exclusive focus on text manifests a conception of proof that has close affinities with the logical conception of proof. As we have seen in § 2.2, a logical proof can also be understood as a text. In the light of considerations like these, we think that views as those expressed by Glymour manifest a way of looking at proofs that is heavily determined by the modern notion of a logical proof (chapter 2).

§ 4.2. Analysis of The Passage

In this section, we set ourselves the task of analyzing the proof described in The Passage. We begin by connecting The Passage with Proclus' framework. Next, we turn to our analysis, again treating the respective items of Proclus' framework successively.

The Passage and Proclus' framework. As noted already, Hintikka [73] has likewise considered Kant's views on the method of proof in mathematics. The novelty of our approach manifests itself on two points. First, our analysis of The Passage goes much deeper than Hintikka's. Second, our conclusions one behalf of our analysis will be somewhat different. (See the concluding remarks at the end of this section.)

Let us now address the following preliminary question: is Proclus' framework the appropriate means in order to analyze The Passage? Somewhat more specifically: is there reason to believe that the method described in The Passage is, at least *grosso modo*, structured in conformity with Proclus' methodological framework? We think that the answer to this question is affirmative. Indeed, this much will become clear *after* we have carried our analysis in the light of Proclus' framework. However, in order to forestall the suspicion that we simply read Proclus' framework *into* The Passage, we need to come up with considerations that are independent of such an analysis.

The most important reason we have for using Proclus' framework is the following: the method described by Kant in The Passage is *grosso modo* similar to the one followed by Euclid in order to prove his theorem 32 from the *Elements*. As we have seen in the previous section, Euclid's method of proof accords with Proclus' methodological framework. Indeed, Proclus set up his framework on behalf of a study of the theorems proved by Euclid in the *Elements*. Consequently, there is ample reason to believe that the method of proof described The Passage also corresponds with Proclus' methodological framework.

Accordingly, our primary justification for using Proclus framework comes with hindsight, by looking backwards into history and noticing certain similarities between Euclid and Kant. Are there also historical reasons for believing that Kant really *accepted* Proclus framework? That is, is there reason to believe that Kant knew of Proclus' framework and that he deliberately structured the proof he described in The Passage in accordance with it? Several strategies are available in order to answer this question.

1. One may try to find an explicit or reasonable explicit reference or allusion to Proclus' methodological framework in Kant's own writings. As far as we can see, however, Kant himself does not make any such reference or allusion. Therefore, this strategy does not appear to bring the matter to an end.
2. We would have reason to believe that Kant accepted Proclus' framework if we could make plausible that it belonged more or less to the commonly accepted background knowledge in Kant's time (e.g., like syllogistic logic did).

Unfortunately, however, it is hard to find good evidence for the claim that Proclus' framework belonged to the common ground of Kant's time. We have found one explicit reference to something very closely akin to Proclus' framework in Leibniz' *New essays on human understanding*.[161] Says Leibniz:

> Geometers start their demonstrations with the 'proposition' which is to be proved, and then prepare the way for demonstration of it by offering the 'exposition,' as it is called, in which whatever is given is displayed in a diagram; after which they proceed to the 'preparation,' drawing in further lines which they need for the reasoning—the finding of this preparation often being the most skilful part of the task. When that has been done, they conduct the 'reasoning' itself, drawing conclusions from what has been given in the exposition and what has been added to it in the preparation; and, with the aid of truths already known or demonstrated, they arrive at the 'conclusion' (Leibniz [102], p.476).

The difference with the framework as presented by Proclus is that in the above citation there is nothing that corresponds to what Proclus calls the *specification*.

The above citation shows that at least some people writing in Kant's time knew of Proclus' framework, or at least something closely akin to Proclus' framework. Moreover, as the editors of the 1996 edition of Leibniz' *New essays* indicate, Kant read the *New essays* in 1769 (Leibniz [102], p.x). This indicates that Kant may very well have known of something closely akin to Proclus' framework.

The upshot of these considerations is the following: on historical grounds, it is as yet not entirely clear that Kant accepted Proclus' methodological framework. Perhaps further historical research will help to decide the matter.

Analysis. It will be convenient to recall The Passage (§ 1.1):

[161] Leibniz published his *New essays* in 1765, though he had a draft already in 1704. For the history of the *New essays*, see Leibniz [102], vii-x.

> He [a mathematician] begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of this triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question (A716-7/B744-5).

Kant characterizes a proof in terms of a certain procedure (§ 1.3), namely, a procedure to the end of establishing the truth of a theorem. The proof Kant describes is a combination of "subroutines," i.e., other procedures. As it turns out, two main types of sub-procedures can be distinguished:

- constructive procedures;
- inferential procedures.[162]

Running the constructive procedures yields the items of knowledge that are needed in the course of the proof. In Kant's view, these items of knowledge are intuitions (§§ 3.2-3), and hence are diagrammatic (§ 3.4). The inferential procedures, in turn, serve to establish the truth of several intermediate conclusions by means of the items of knowledge created (how this is done will become clear later). Accordingly, the reasoning always proceeds *in concreto* (§ 3.3).

We can look at inference in two different ways, namely from a "static" point of view and from a "dynamic" point of view. From the first, static point of view, the inference is seen in terms of the grounding relations among several propositions. From the second, dynamic point of view, the inference turns on items of a diagrammatic format. Since the point will reoccur for several times later, it is useful to illustrate it here and to introduce some convenient terminology.

First, consider the following definitions:

DEFINITION 1. A triangle is a plane figure contained by three lines.

DEFINITION 2. A right-angled triangle is a triangle having one right angle.

Consider also the following theorems:

[162] See Lindsay [105], [106], for a striking parallel with more recent theorizing about inference in cognitive science.

THEOREM 1. Given a line, there exists a line through any point not on that line which is perpendicular to it.

THEOREM 2. If in a right-angled triangle a perpendicular were drawn from the right angle to the base, the triangles adjoining this perpendicular are congruent to this right-angled triangle and to one another.[163]

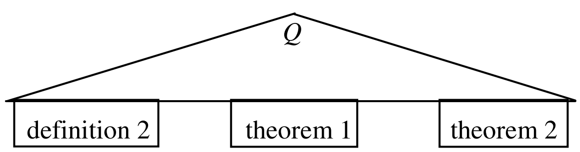
Note that, in case of theorem 2, the perpendicular indeed exists. This is secured by theorem 1. Now consider the following inference.

Let *ABC* be a right-angled triangle having $\angle ABC$ right. Let *AD* be a line perpendicular to the base side *BC*. We can infer the following:

Q: the triangles *ABD* and *ADC* are congruent to one another and to triangle *ABC*.

Now, the point we want to bring out concerns two different but intimately related inferential roles of definition 2, theorem 1 and theorem 2 in relation to the inferring of *Q*.

First, we may say that the conclusion *Q* is *based on* definition 2, theorem 1, and theorem 2. This concerns primarily the "static" grounding relation between definition 2, theorem 1, theorem 2 on the one hand and *Q* on the other. This relation is made vivid below:



Note that it is reasonable to assume the following: a known proposition based on propositions known a priori is itself a priori. Since, according to Kant, all the known propositions of mathematics are a priori, we can conclude that *Q* is also a priori.

To the above diagram, we can straightforwardly associate the following simple inference:

$$\frac{\text{definition 2, theorem 1, theorem 2}}{Q.}$$

[163] Cf. Euclid [41], I, p.154, p.270; *ibid*., II, p.209.

The impression arising is the following. When we think of inferring Q in terms of a simple inference, then we have our eye first of all on the grounding relations among several propositions (i.e., definition 2, theorem 1 and theorem 2 on the one hand and Q on the other). It is accordingly that the inference leading to Q gets a strong propositional flavor: the inference is in effect a simple inferential transition from a number of propositions to another proposition.

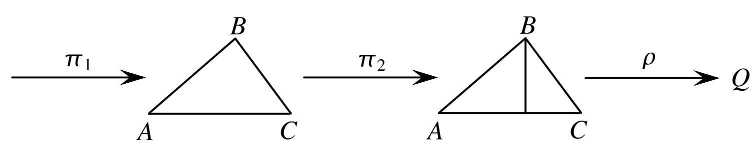
Second, instead of first considering the aforementioned grounding relation, we may alternatively concentrate on the structure of the inferential procedure delivering Q as its product. This leads to a different picture. Let us explain.

The inferential procedure we are currently considering can be naturally seen as a sequencing of the following three sub-procedures.

π_1: construct a right-angled triangle ABC;
π_2: construct a line through the point B, perpendicular to AC;
ρ: infer the conclusion.

Carrying out π_1 and π_2 in sequence yields a composite constructive procedure we may write as $\pi_1;\pi_2$. (We let ';' denote a (binary) sequencing operation on procedures.)

The establishment of Q's truth is the product of (a run of) another composite procedure. Assuming that the operation ';' is associative, we may write this procedure as $\pi_1;\pi_2;\rho$. However, along the way, two intermediate items of knowledge (i.e., intuitions) are also produced. Together, the procedure $\pi_1;\pi_2;\rho$ may be traced as follows:



In case of the specific example we are currently considering, the procedural role of definition 2, theorem 1 and theorem 2 is somewhat as follows. Definition 2, theorem 1 and theorem 2 together *guide* the execution of the procedures π_1, π_2 and ρ respectively. Specifically, we may say that a run of π_1 is *in accordance with* definition 1. Similarly, we may say that a subsequent run of π_2 is in accordance with theorem 1. Finally, we may say that a run of ρ is in accordance with theorem 2. What we see is that from this procedural point of view, the inference turns out to be made up of items of a different format, viz. diagrammatic instead of propositional.

Note that that logical notions such as validity and soundness straightforwardly apply when we think of the inference as a simple inference. However, it is not so easy to see how these notions apply when we focus primarily on the structure of the inferential procedure. In the first case, we may say that the truth of Q is secured via the validity of the inference and its soundness. In the second case, in contrast, the truth of Q is secured via a procedure that is in some sense valid and sound at once. In other words, Q's truth is not primarily secured via logical properties such as validity and soundness, but via the specific structure of the process delivering Q as its product (cf. § 2.1).

Note also that the establishment of Q's truth is not merely the product of a run of the inferential procedure ρ. In contrast, establishing Q's truth involves an execution of the constructive procedures π_1 and π_2 as well. In other words, the establishment of Q's truth comes as the product of running a *constructive-inferential procedure*, which is precisely the composite procedure $\pi_1;\pi_2;\rho$.

However, this inferential procedure, as Kant sees it, does not refer to logical rules of inference of an underlying logical system. In contrast, it refers to the axioms, definitions and theorems of an underlying *mathematical theory* (or science). In particular, these axioms, definitions and theorems are not propositional items reasoned *with*. The items reasoned with are themselves primarily diagrammatic. On the other hand, the mathematical theory constituted by the axioms, definitions and theorems serves as a kind of background knowledge that can be utilized in order to carry out inferences. From a procedural point of view, then, the inferential regime of the inference does not seem to be a logical one, but primarily a mathematical one. Let us now turn to a detailed analysis of The Passage, treating the six parts of Proclus' framework successively (cf. also § 1.1).

1. The enunciation. Recall that, according to Proclus, the enunciation concerns something given and a goal ("what is sought"). In the light of this, we suggested that the enunciation appears to set forth a certain task or problem (§ 5.1). This suggestion becomes actually quite plausible in the case we are currently considering. In order to see this, let us take a look at the wider context from which The Passage is quoted.

Kant describes how he thinks a mathematician proves theorem 1 from § 1.1 in a context where he contrasts the philosopher's use of reason with the mathematician's use of reason.[164] Kant says:

> Give to a philosopher the concept of a triangle, and let him discover in his own way what the relation of the sum of its angles to a right angle might be (A716/B744).

[164] The Passage is intended to exemplify the typical mathematician's use of reason.

Kant goes on by pointing out that the philosopher comes not further than analyzing the concept of a triangle into, for example, the concept of a straight line, or an angle, or of the number three (cf. *ibid*.).[165] Then he proceeds by saying: "[b]ut now let the geometrician take up this question" (*ibid*.). Subsequently, The Passage follows.

Clearly, Kant sees The Passage as an answer (or solution) to a certain question. This is further confirmed by the final words of The Passage: "[…] he arrives at a fully illuminating and at the same time general solution of the question." As a question can be naturally seen as a kind of problem,[166] we can conclude that in The Passage, Kant describes a mathematician who is busy solving a problem. Note that Kant accordingly interprets theorem proving in a quite specific way, namely, as a species of *problem solving*.

Given the wider context from which The Passage is taken, it seems plausible to hold that what is given in case we are currently considering is (the concept of) a triangle. Furthermore, the goal is to find the proportion between the sum of its internal angles and a right angle. As will be expected, this proportion is 1 : 2.[167] Equivalently, then, the goal is to show that the sum of the internal angles is equal to two right angles.

2. The exposition. Kant says: "He [i.e., a mathematician] begins at once to construct a triangle." In Kant's view, the exposition concerns the execution of a constructive procedure. For Kant, to construct a triangle means to exhibit an intuition corresponding to the concept `triangle`. We have seen that an intuition is a diagrammatically formatted item of knowledge. In order to fix our thoughts, then, let us assume that he exhibits this intuition in terms of the following diagram (which we henceforth refer to as *diagram I*):

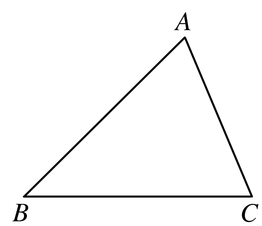


Diagram I

In order to facilitate later cross-referencing and identification, we have added the labels *A* and *B* and *C* to the diagram (cf. Euclid's diagram from § 4.1; for more

[165] What Kant has effectively in mind is presumably that the philosopher comes no further than analyzing the concept of a triangle into the defining properties of a triangle (cf. A718-9/B746-7).
[166] See Hintikka [77].
[167] That is, one times the sum of the internal angles stands to two times a right angle.

on labels, see also § 4.4). There is no evidence that the mathematician added these labels himself. Nor, by the way, is there any evidence that shows that he didn't.

Prima facie, there is even no need for this mathematician to identify an angle or side by means of labels. For example, he may identify an angle or side by means of its relative position with respect to other diagrammatic objects. Furthermore, the mathematician may have wished to add a label if he intended to *communicate* the proof he is currently carrying out. For example, when he was writing a paper by means of which he intended to communicate the proof to a third party, or when he was proving the theorem stated in front of a group of students. However, considerations like these go far beyond the purposes of the present work.

Diagram I comes as the outcome of what is in Kant's view a distinctly mathematical procedure, viz. a constructive procedure. Consequently, the mathematician's constructive activities are in Kant's view already operative in the exposition. This appears to be a difference with Proclus. For the latter did not explicitly speak of construction in case of the exposition (§ 4.1).

It would seem that a mathematician cannot exhibit an intuition in terms of any diagram he likes. For example, in the case we are currently considering, he cannot exhibit an intuition in terms of the following diagram:

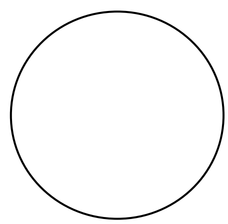


Diagram I.a

Again, to construct a concept means: to exhibit an intuition corresponding to that concept. What could *corresponding* mean under such a reading?

We think that part of the answer is that diagram I.a does not "correspond" to the concept of a triangle. For example, Euclid defines a triangle as a rectilinear figure enclosed by three straight lines (see definition 1 above; cf. Euclid [41], I, p.154). In contrast, Euclid defines a circle as a figure enclosed by one line such that all the points on that line are equidistant from a given point (the *center* of the circle) (cf. *ibid*., p.153). Accordingly, diagram I.a corresponds to a circle rather than a triangle.

3. The specification. We would expect that the specification, if Kant would have mentioned it in The Passage, is a statement to the effect that the sum of the angles $\angle ABC$, $\angle BCA$, $\angle CAB$ is equal to two right angles (cf. diagram I). However, there appears to be nothing in The Passage that can be naturally interpreted as corresponding to such a specification.

As a possible alternative for the specification, one might consider the mathematician's knowledge "that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line." However, a little reflection on The Passage shows that this is not properly speaking the goal that needs to be reached. In contrast, it is what motivates the proof. Let us conclude, then, that The Passage contains no explicit specification.[168]

Note that Kant does not seem to choose his words very carefully when he describes the knowledge that motivates the proof. Kant says: "that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line." What Kant evidently has in mind is that all of the adjacent angles to be drawn lie on one side of the line.[169] Put differently, what Kant evidently has in mind is:

THEOREM 3. Two right angles together are equal to all the angles that can be drawn at one point on a straight line in such a way that all these angles lie on the same side of that line.

Theorem 3 is evidently a theorem of Euclidean geometry. However, it is not stated as such in Euclid's *Elements*. The theorem it comes most close to is theorem 13 from the first book of the Elements (see § 4.1). In a sense, theorem 3 is more general then theorem 13.

4. The construction. During the construction, the mathematician carries out two extra constructive procedures on top of the one carried out in the exposition. In particular, the construction involves the addition of two auxiliary lines to diagram I. Thus, diagram I goes through two successive modifications.[170] Let us consider them successively.

As to the first constructive procedure, Kant says: "[h]e extends one side of this triangle [i.e., the one exhibited in the exposition]." To fix our thoughts, suppose the mathematician extends the base side of the triangle involved in the exposition to a point we call *D*, as follows:

[168] See also the quotation from Leibniz' *New essays*, cited in § 4.1.
[169] We already made this point in footnote 6.
[170] This more or less matches with the development of the diagram seemingly involved in Euclid's proof of theorem 32.

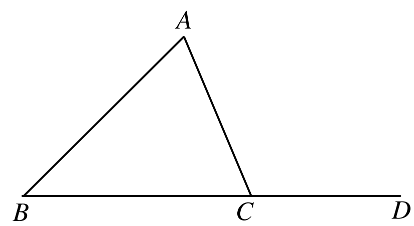


Diagram II

As to the second constructive procedure, Kant says that the mathematician "divides the external one of these angles by drawing a line parallel to the opposite side of the triangle". The external angle Kant talks about is precisely $\angle ACD$. Again, to fix our thoughts, let us assume that he divides this angle by drawing a line *CE*, as follows:

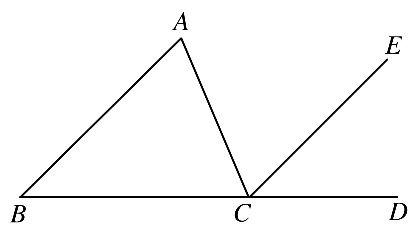


Diagram III

Let us make some remarks on behalf of the two steps that have been undertaken in the construction.

a. As in the case of diagram I, the labels *D* and *E* are primarily added by *us*. Similar remarks apply as in the case of diagram I (see above).

b. When the mathematician extends a side of the triangle exhibited in terms of diagram I to the point we have called *D* in diagram II, we may say that he has executed a constructive procedure in accordance with Euclid's second postulate (see § 4.1). Accordingly, this postulate functions as a kind of methodological rule (or principle) guiding a constructive procedure.

Similar points can be made of the line *CE* added to diagram I (cf. diagram III). When the mathematician adds a line parallel to the opposite side (cf. ii. above), we may say that he carries out a constructive procedure *in accordance with* theorem 31 (see § 4.1). Again, on this particular occasion, theorem 31 functions as a kind of methodological rule (or principle) guiding a constructive

procedure. Collectively, diagram III comes as the result of successively carrying out three constructive procedures.
c. Let us also note that the mathematician whose methods Kant is describing did not draw the additional lines without purpose. Quite the contrary, Kant says that the mathematician did so in view of his knowledge of theorem 3. This knowledge seems to be what motivates the subsequent procedures to be carried out. Accordingly, these procedures appear to be executed in a strict purposeful manner. The impression arising is that the task or problem the mathematician was confronted with in the enunciation evokes a kind of "programmed response" (as we may call it).
d. Constructive procedures also come into play in case of the exposition. Note that this seems to be a difference with what Proclus himself appears to suggest. For according to the latter, a mathematician's constructive activities only seem to play a role in that item of Proclus framework called the *construction*.

Before we turn to the *apodeixis*, it will be convenient to introduce some notational abbreviations.

We refer to the constructive procedure resulting in diagram I as κ_1. The constructive procedure that modified diagram I, thus resulting in diagram II, is referred to as κ_2. Finally, the constructive procedure modifying diagram III, thus resulting in diagram III, is referred to as κ_3.

We may now say that diagram II comes as the result of a certain composite constructive procedure, namely, the procedure that consists of carrying out κ_1 and κ_2 in sequence, that is, as a composite procedure $\kappa_1;\kappa_2$.

Furthermore, diagram III comes as the result of another composite constructive procedure, namely, the result of sequencing the procedures $\kappa_1;\kappa_2$ and κ_3, i.e., $\kappa_1;\kappa_2;\kappa_3$. The three relevant constructive procedures and their respective outputs are summarized below:

- $\kappa_1 \longrightarrow$ diagram I
- $\kappa_1;\kappa_2 \longrightarrow$ diagram II
- $\kappa_1;\kappa_2;\kappa_3 \longrightarrow$ diagram III

It will be convenient to have these notations for the sake of future reference.

Before we end our discussion of the construction, let us first make clear that the mathematician Kant describes in The Passage strictly does as a matter of fact not carry out the procedure $\kappa_1;\kappa_2;\kappa_3$, that is, he does not carry out the procedures κ_1, κ_2, and κ_3 in a row. In particular, right after the procedure κ_2, the mathematician does not go straight on to execute κ_3. In contrast, he first executed an inferential procedure, delivering the establishment of the truth of a conclusion as a product (see below). Only then he proceeds to κ_3. We shall return to this later.

Diagrams I, II, and III can be naturally seen as three successive stages of a *dynamic* diagram (§ 3.4), that is, a diagram that evolves through (discrete) time. Thus, the intuition or intuitions that play a role in the course of a proof are not static. In contrast, they evolve through time. This may seem a remarkable point. In Kant's view, the knowledge a mathematician reasons with (in the form of intuitions) seems not static but dynamic instead. In Kant's view, the evolvement of intuition turns precisely on the workings of a specific creative force, viz. the productive imagination (§ 3.4). Thus, proving a theorem can be quite properly typified as a procedure of proof *creation*.

5. The apodeixis. The *apodeixis* concerns the inferential procedures that are executed in the course of a proof. In The Passage, we find an explicit description of two inferences. The first is where Kant says that the mathematician "obtains two adjacent angles that together are equal to two right ones." The second is where Kant says that the mathematician "sees that there arises and external adjacent angle which is equal to an internal one." Kant does not explicitly state the other inferences that need to be drawn in order to arrive at the desired result. We made a plausible suggestion as two what those inferences might be in § 1.1. It will convenient to repeat them here.

First, the mathematician sees yet another external angle that is equal to an internal one.

Second, he seems to collect what is obtained thus far, and concludes that the sum of the internal angles is equal to two right angles. (See below for clarification.)

Collectively, the mathematician carries out four inferential procedures—or so we assume. The product of any of these inferential procedures is the establishment of the truth of a conclusion, among which is the theorem itself. Let us discuss these inferences successively. We pay most attention to the first inferential procedure. Many of the points raised there will also apply to the others.

i. The first inference. The first inference is executed at the point where Kant says that the mathematician "obtains two adjacent angles that together are equal to two right ones." Let us first state more clearly what the conclusion is that is being drawn.

Note that the conclusion—or better: Kant's formulation of this conclusion—makes mention of a relation, namely, a relation of *adjacency* between two angles. In the present context, this relation should be evidently understood as a diagrammatic relation, i.e., a relation in diagrammatic space. This relation determines how two angles are located with respect to one another in diagrammatic space.

It seems reasonable to believe that the relation of adjacency mentioned (by Kant) is in fact a specific constituent of diagram II, namely, the relation of adjacency that holds between $\angle ACB$ and $\angle ACD$. What the mathematician more specifically concludes, then, is that two adjacent diagrammatic objects—viz. $\angle ACB$ and $\angle ACD$—are together equal to two right angles. In other words, the mathematician establishes the truth of a conclusion we may write as follows:

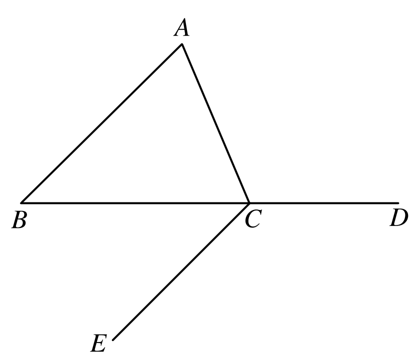
Q_1: $\angle ACB + \angle ACD = 180°$.[171]

Let us refer to the inferential procedure we are currently considering as σ_1. Let us consider in somewhat more detail how the truth of Q_1 is established.

The truth of Q_1 has not been established merely by way of running the inferential procedure σ_1. In contrast, the mathematician establishes Q_1's truth by way of an execution of the composite procedure $\kappa_1;\kappa_2;\sigma_1$, that is, the sequencing of the (composite) constructive procedure $\kappa_1;\kappa_2$ and the inferential procedure σ_1. In Kant's view, then, the mathematician comes to know Q_1 as the product of what we may properly call a *constructive-inferential procedure*.

Recall that the procedure κ_1 has been executed in accordance with the definition of a triangle and that κ_2 has been carried out in accordance Euclid's second postulate. Furthermore, the inferential procedure σ_1 is carried out in accordance with theorem 29. (See above.) Collectively, then, the definition of a triangle, Euclid's second postulate and theorem 29 function as methodological principles guiding the execution of the procedure $\kappa_1;\kappa_2;\sigma_1$.

Note that, when Euclid added line *CE* to the first diagram, he did it in such a way that point *E* turned out to be located *above* line *BC*. However, it is interesting to consider what would happen if he had drawn line *CE* in such a way that point *E* turned out to be located *below* line *BC*. All other things being equal, this would result in a diagram that looks, say, as follows:



[171] Recall that 180° is equal to two right angles (cf footnote 3).

This time, too, CE is parallel to AB and AC is intersecting both AB and CE. However, it does *not* follow that $\angle BAC = \angle ACE$. It seems that we are led to conclude that the truth of the conclusion drawn depends on the specific diagrammatic relations (i.e., relations in diagrammatic space) among diagrammatic objects. In particular, the conclusion depends on the relative position of the point E with respect to the line AB.

ii. The second inference. The second inference is drawn at the point where Kant says that the mathematician "sees that there arises an external adjacent angle which is equal to an internal one."

Note again that this conclusion mentions several spatial relations First, there is mention of a relation of *adjacency* between two angles. Second, there is mention of an angle that is *internal* with respect to the triangle constructed. Third, there is mention of a relation that is *external* with respect to the triangle constructed. Again, using terminology introduced earlier, these relations hold among certain diagrammatic objects in a diagrammatic space.

We can state the second inference more clearly using diagram III (see above). What Kant is trying to say can accordingly be rephrased as follows: there arises an external adjacent angle $\angle ACE$ (say) equal to the internal angle $\angle BAC$. In other words, the mathematician establishes the truth of a conclusion we may write as follows:

Q_2: $\angle ACE = \angle BAC$.

Henceforth, we shall refer to the inferential procedure that delivers the establishment of Q_2's truth as its product as σ_2.

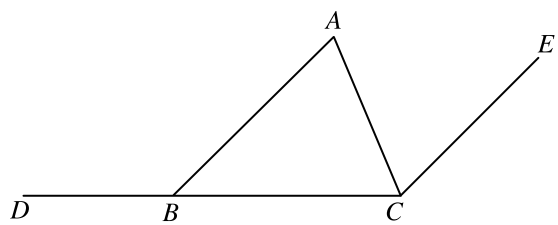
The same questions may be asked as in the case of the inferential procedure σ_1. First, how does he carry out the procedure σ_2? Second, what constitutes the basis of Q_2? The respective answers to these questions proceeds in a similar way as above.

As to the first question, note that Q_2's truth has not been established merely by way of an execution of the procedure σ_2. In contrast, the mathematician comes to know Q_2 as the product of the composite procedure $\kappa_1;\kappa_2;\kappa_3;\sigma_2$. As in the case of the first inference, we may refer to thus composite procedure as a constructive-inferential procedure.

Note that σ_2 is an inferential procedure carried out in accordance with theorem 29. Recall that the constructive procedure κ_3 is carried out in accordance with theorem 31 (see above). Collectively, the definition of a triangle, Euclid's second postulate, theorem 31, and theorem 29 function as methodological principles guiding the execution of the procedure $\kappa_1;\kappa_2;\kappa_3;\sigma_2$ delivering the establishment of Q_2's truth as its product.

As a further observation, suppose Euclid would have drawn line *CE* in such a way that point *E* turned out to be located below line *BC* (cf. the diagram above). Then, all other things being equal, *AB* is still parallel to *CE* and *BD* intersects both *AB* and *CE*. However, it would not be true that $\angle ECD$ equals $\angle ABC$. As in the case of the first inference, the truth of the conclusion would seem to be dependent on a specific spatial relation among (other) diagrammatic objects. Therefore, it would again seem that we are entitled to conclude that inference (2) is not a logical deduction from a true premise.

Furthermore, suppose that Euclid would have extended the line *BC* to *D* in the opposite direction, that is, in such a way that *D* turns out to be located to the left of *B* rather than to the right of *C*. All other things being equal, this would result in a diagram that looks, say, as follows:



In this case, *AB* would still be parallel to *CE*. It does not seem, however, that line *BD* still intersects line *CE*, unless *BD* continues to at least the point *C*. Furthermore it is neither true that $\angle ECD = \angle ABC$.

What we see is that each of the diagrams I-III organizes the knowledge used in the proof in a distinctly spatial way. Accordingly, in Kant's view, diagrams are not merely psychological aids. Instead of using relational expressions such as *lies above*, *lies adjacent to*, *lies opposite to*, etc., a mathematician in Kant's view exhibits these relations as relations in space, i.e., in terms of diagrammatic relations (§ 3.4). These relations are *employed* in the course of the proof. This is precisely what makes the reasoning genuinely diagrammatic.

iii. The third inference. The third conclusion is drawn where the mathematician concluded that there arises yet another external angle that is equal to an internal one (or so we assumed). As in the case of the previous two inferences, similar remarks can be made on behalf of the diagrammatic relations mentioned. In short, the mathematician establishes the truth of the following conclusion:

Q_3: $\angle DCE = \angle ABC$.

Further discussion of this inferential procedure would add no substantially new points. Therefore, we proceed directly to a discussion of the final inference.

iv. The fourth inference. The impression arising is that the fourth inference more or less collects the results obtained thus far. More specifically, the impression may arise that the mathematician now concludes:

Q_4: $\angle ABC + \angle BCA + \angle BAC = 180°$.

from Q_1, Q_2, and Q_3. In other words, the fourth inference appears to be a simple inference (§ 2.3):

$$\frac{Q_1, Q_2, Q_3}{Q_4.}$$

An alternative reading is possible, however, a reading that puts the fourth inference on a par with the previous three constructive-inferential procedures.

Perhaps an inclination to consider the fourth inference as a non-constructive inference comes from the fact that Q_1, Q_2 are described by Kant as two intermediate conclusions. No doubt, this is something could also be said of Q_3, had Kant offered a full description of the proof of § 1.1, theorem 1, in The Passage. Accordingly, the impression may arise that the mathematician whose proof Kant is describing himself likewise drew these conclusions. We may then go on by saying that this mathematician stored Q_1, Q_2, and Q_3 somewhere until he needed them. At that point, he retrieved Q_1, Q_2, and Q_3 in order to draw Q_4 as a conclusion from them.

However, we must not too readily conclude something about the structural organization of a process from Kant's *description* of it. In this respect, it need not be the case that the mathematician's conclusion Q_4 forms the product of (a run of) a certain non-constructive-inferential procedure, namely, one from the premises Q_1, Q_2 and Q_3. In contrast, it may very well be that Q_4 forms the product of the entire constructive-inferential proof procedure as it has been executed thus far. This reading accords well with the constructive outlook of the proof. Without definitely resolving the issue, let us briefly speculate a bit further on the issue.

Again, reconsider diagram III above. Before Q_4 was drawn as a conclusion, we may say that the mathematician conceived of this diagram in a specific way. In particular, it seems not unreasonable to say that he saw the diagram mainly as a configuration of *angles* such that some of these angles coincided with others (e.g., $\angle ACE = \angle BAC$, $\angle DCE = \angle ABC$). At the point he draws Q_4 as a conclusion, in contrast, he primarily sees it as a *triangle* (plus, perhaps, some auxiliary lines) whose internal angles sum up to 180°. In other words, his intentional attitudes with respect diagram III have changed, and hence diagram III itself has changed (cf. § 3.4).

Perhaps this change in intentional attitudes with respect to a diagram can itself be interpreted as a constructive step. If we refer to the attitude change we are currently considering as κ_4, then we may say that the inferential procedure delivering Q_4 as its product at least involves the constructive procedure $\kappa_1;\kappa_2;\kappa_3;\kappa_4$. In a way, then, the procedure leading up to the establishment of Q_4's truth is constructive in its entirety.

6. The conclusion. Kant clearly alludes at something that can be naturally interpreted as a conclusion when he says: "In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question." (Note that Kant sees the conclusion in terms of a solution of a question.)

One way to understand this is as follows. What Kant has described in The Passage is a mathematician proving a theorem by way of broadly the following method. The proof mainly proceeds in terms of a consideration concerning one *specific* triangle (this triangle is introduced in the construction). What the mathematician has shown is that the sum of the internal angles of *this* triangle is equal to two right angles. Now, when this mathematician finally arrives at the conclusion (or, as Kant says, arrives at his solution), he carries out a procedure of universal generalization to the effect that *all* triangles have the property mentioned. This is what Kant would have had in mind when he says that the mathematician arrives at a "*general* solution of the question" (emphasis added). It is primarily the conclusion that is general (i.e., concerns all triangles). The method that in the end leads to this general conclusion mainly concerns an individual specific triangle and is in that sense itself specific. This is essentially the way Beth [16], [18] (especially chapter 4) and Hintikka [72], [73] interpret the proof Kant's describes in The Passage.

However, it is not entirely clear that Kant sees the proof as proceeding this way. It is not clear that Kant thinks of the proof as mainly proceeding in terms of a consideration that concerns a *specific* triangle and in the end generalizes to *all* triangles (by way of an extra step of universal generalization). Notice that when Kant says that the mathematician arrives at a general solution to the question, he refers to the *way* this solution has been obtained: "[i]n *such a way*, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question." The "way" Kant is referring can be naturally interpreted as the (mainly constructive) procedures that have been executed thus far. Accordingly, the generality primarily applies to the *method* of proof (cf. A714/B742).

In this respect, then, there seems to be no point in saying in the conclusion that all triangles have the property in question, when the latter is meant as the product of an extra inferential step (i.e., a step of universal generalization).[172] It may very well be that Kant thinks that the procedures that have been executed in the exposition, the construction and the *apodeixis* do precisely this: together, they prove that all triangles are such that their internal angles add up to two right angles.

The last point accords well with the following. Kant says that

> […] mathematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless *a priori* and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept […] must likewise be thought as determined universally (A714/B742).

The individual in terms of which mathematics contemplates the general is precisely an intuition (cf. *ibid*.). In some way, then, such an individual intuition makes a mathematician "see" the general. This may sound rather mysterious, but upon closer inspection, there appears to be an interesting idea underlying it. Let us explain.

Besides the fact that mathematics contemplates the universal in the particular, Kant also says that mathematics "considers the concept *in concreto*" (A715/B743). Put differently: "mathematics contemplates the concepts only in an intuition that it has exhibited a priori [i.e., constructed]" (A716/B744). In our view, this makes it not entirely clear that in case of the proof we are currently considering, an intuition must be understood as representing a *specific* triangle. In contrast, especially the last two remarks of Kant's suggest that, in some sense, this intuition is closely akin to the *concept* of a triangle. However, it should be emphasized that, in Kant's view, this intuition *is* not the concept of a triangle; it is the product of constructing that concept. Neither should we believe it to be an instantiation of this concept (i.e., an individual triangle). In contrast, it is the concept of a triangle made concrete.

The idea that the proof Kant describes in The Passage proceeds mainly on the basis of a consideration concerning a specific triangle and then generalizes to all triangles corresponds well with certain methods of proof in first order logic (§ 2.3). In this respect, the introduction rule for the universal quantifier plays an important role. It may very well be that Beth's and Hintikka's respective readings of The Passage (cf. above) are to a considerable extent influenced by insights stemming from standard proof procedures from first-order logic. However, the considerations above indicate that an alternative reading must certainly not be excluded. More positively, we think that the reading sketched above may very well amount to a more plausible interpretation of Kant.

[172] One is reminded of Proclus' remark that mathematicians "are accustomed to draw what is in a way a *double* conclusion" (see § 4.1; emphasis added).

Concluding remarks. We state our conclusions on behalf of our analysis of The Passage.

1. When a mathematician proves a theorem, he, among other things, carries out several constructive-inferential procedures. The constructive procedures that in part constitute them are carried out in that part of Proclus' framework called the exposition, but also in the part referred to as the construction. Hence, the inferences doe not merely concern the *apodeixis*; as constructive-inferential procedures, they concern the exposition and the construction as well. Accordingly, in Kant's view, the entire proof is permeated with constructive elements.
2. We found that several mathematical axioms, definitions and theorems often function as a kind of rules (or principles) in the course of proving a theorem. Consequently, we may say that the inferential regime is to a considerable extent constituted by axioms, definitions and theorems. The items reasoned with, furthermore, are mainly intuitions. This idea is hard to square with the modern idea of a logical system. From the latter point of view, the axioms and theorems on the one hand and the rules of inference on the other, from two separate categories. The inferential regime is constituted by the rules of inference; the items reasoned with are axioms, theorems, and other propositional items such as assumptions. See chapter 2 for details. In Kant's view, in contrast, axioms, definitions and theorems have a more flexible use. On the one hand, they are the propositional items of knowledge that constitute mathematical science. On the other hand, however, they can also form the inferential principles of a mathematical proof.
3. Intimately related to our second conclusion, we argued that the items of knowledge figuring in the proof—i.e., the intuitions a priori—are not propositional but diagrammatic instead. Since the diagrammatic features of those items where really employed in the course of the proof, we may conclude that in Kant's view, proving a mathematical theorem is (essentially) a form of diagrammatic reasoning.
4. As pointed out in § 2.3, Beth and Hintikka hold that Kant's views on mathematical proof can be adequately characterized in terms of modern systems of natural deduction. From a procedural point of view, we can now say that this amounts to a mischaracterization of the structural organization of a proof process. According to Kant, a mathematician reasons mainly with diagrammatic items of knowledge. According to a view as put forward by Beth Hintikka, in contrast, a mathematician would reason with primarily with propositional items instead.
5. We can now confirm a point that was raised in § 2.3. Recall that Beth associates the constructive character of a Kantian mathematical proof to the introduction rule of the universal quantifier. Hintikka, in contrast, concentrate on those rules of natural deduction according to which new individuals are introduced in the course of a proof. This implies that Hintikka associates the constructive nature of a Kantian mathematical proof with the elimination rule for

the universal quantifier and the elimination rule for the existential quantifier. Hintikka focuses especially on the elimination rule for the existential quantifier.[173]

We now clearly see that if there is any logical inference corresponding to what is going in a proof according to Kant at all, it must be an annotated logical inference (of which the elimination rule for the existential quantifier is only one; § 2.3). According to Kant, in order to draw a conclusion, it is essential that a constructive procedure be carried out first. Something similar happens in the case of annotated logical inferences. For we noticed that in the case of such logical inferences, in order to draw a conclusion, it is essential that a derivation be produced first.

However, as we observed (§ 2.3) the fact remains that logical proof can be entirely formulated in terms of a language (sentences). This, we think, forms one great difference with Kant's conception of inference as exemplified in The Passage.[174] We may say that in case of the proof described in The Passage, language does not seem to play any significant role. Perhaps a Kantian mathematical proof can be reconstructed in terms of a logical proof, and hence in terms of a language. However, such a reconstruction would not seem to be able to reflect certain procedural characteristics that are, in Kant's view, essential for a mathematical proof. The most important of these procedural characteristics is precisely the essential constructive nature of a mathematical proof procedure.

§ 4.3. Conclusion

In the present chapter, we have analyzed The Passage in the light of Proclus' methodological framework for proving theorems in mathematics. According to this framework, one (ideally) proves a theorem by way of an enunciation, an exposition, a specification, a construction, an *apodeixis*, and a conclusion. Considering The Passage, many of these items of Proclus' framework can be naturally discerned. According to Kant, both the exposition and the construction are constituted by the execution of several constructive procedures. In the exposition an intuition is produced, in the construction it is modified for several times. Furthermore, the inferential procedures carried out the *apodeixis* essentially involve the constructive procedures executed in the exposition and the construction. These constructive procedures deliver a diagrammatic item of knowledge (an intuition a priori) as their product. Hence, it became appropriate to refer to these inferential procedures as constructive-inferential procedures.

---

173 An application of these rules, Hintikka holds, correspond to what he calls "synthetic inference steps" (Hintikka [76], p.136).

174 Another difference turns of the non-locality of logical inferences; in Kant's view, in contrast, mathematical inferences are highly local—i.e., mathematical—in nature (cf. also § 2.3).

Collectively, in Kant's view, a proof can be typified as being mainly *constructive* (in the sense of Kant). Related to this, proving a theorem is, in Kant's view essentially a form of diagrammatic reasoning.

Chapter 5

The synthetic a priori in mathematics reconsidered

In the present chapter, we want to turn our insights on two further themes. The first concerns Kant's famous claim that the propositions of (pure) mathematics—and hence the theorems of mathematics—are synthetic and a priori. The second theme concerns Kant's views on the relation between the methodology of mathematical proof and logic.

In § 5.1, we discuss Kant's distinction between analytic and synthetic theorems. In § 5.2, Kant's views on the methodology of proving analytic propositions are explained in some detail and compared with the methodology of mathematical theorems (cf. § 4.2). Finally, § 5.3 addresses the relation between logic and methodology, and the relation between logic and the methodology of mathematical proofs in particular. As always, we end up by stating our conclusions (§ 5.4).

The previous paragraph indicates that we will pay at least some attention to Kant's views on analytic propositions (despite the fact that the theorems of mathematics are synthetic). Our motivation to do so stems from two sources. First, the nature of synthetic propositions (and synthetic a priori theorems in mathematics in particular) will be better understood when we also consider the contrasting class of analytic propositions. Second, and related to this, we will see that an important motivation for Kant to distinguish between analytic and synthetic propositions turned on certain specific cognitive issues.

§ 5.1. Analytic and synthetic propositions: definitions and criteria

In the present section, we discuss Kant's views on the distinction between analytic and synthetic propositions. We only consider true propositions. We begin by squaring the analytic-synthetic distinction with another distinction Kant has made, namely, the distinction between a priori and a posteriori (see § 3.2). Subsequently, we consider how Kant defines analytic and synthetic propositions. In the next two sections, we discuss the respective criteria for analytic and synthetic propositions. The relevant terminology will be clarified as we go along.

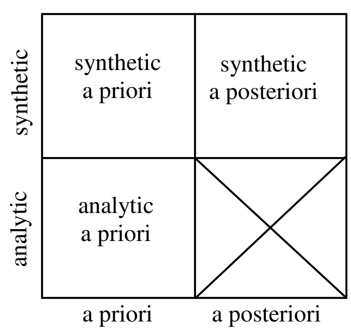
Kant's division of propositions. As said in the introductory paragraphs to this chapter, Kant famously claimed that all the propositions of (pure) mathematics are without exception synthetic and a priori:

> *Mathematical judgments are all synthetic.* […] It must first be remarked that properly mathematical propositions are always *a priori* judgments and are never empirical, because they carry as necessity with them, which cannot be derived from experience. But if one does not want to concede this, well then, I will restrict my proposition to *pure mathematics*, the concept of which already implies that it does not contain empirical but merely pure *a priori* cognition (B14-15; cf. Kant [95], p.63).

A proposition in mathematics is an axiom, a definition or a theorem (A726/B754). When Kant says that all propositions of mathematics are synthetic and a priori, what he therefore means is that the axioms, definitions and the theorems of mathematics are synthetic a priori (cf. A727-33/B755-61; A10/B14). We are mainly interested in the theorems of Euclidean geometry.

Besides the a priori ones, synthetic propositions also come in the a posteriori variety. Thus, Kant clearly suggests that all "experiential" propositions are synthetic and a posteriori (A7-8/B11-2; cf. also A9/B13). Experiential propositions can be quite naturally associated with what we would nowadays call empirical or natural science (cf. A7-8/B11-2).

Kant held that analytic propositions are always a priori; there are no analytic a posteriori propositions. Collectively, then, Kant acknowledged three types of propositions. They are depicted in the diagram below:



Insofar as the purposes of the present study are concerned, we are exclusively interested in propositions of mathematics, which live in the upper left part of the above diagram. Nevertheless, it will prove fruitful if we also consider analytic propositions, which live in the lower left part.

Analyticity and syntheticity defined. Kant defines analytic and synthetic propositions as follows:

> In all judgments in which the relation of a subject to the predicate is thought (if I consider only affirmative judgments, since the application to negative ones is easy), this relation is possible in two different ways. Either the predicate *B* belongs to the subject *A* as something that is (covertly) contained in this concept *A*; or *B* lies entirely outside the concept *A*, though to be sure it stands in connection with it. In the first case I call the judgment analytic, in the second synthetic (A6-7/B10; cf. Kant [95], p.62).

Kant's definition of analytic as well as synthetic propositions is stated in exclusively terms of certain relations that hold between concepts. In particular, Kant does not consider quantifiers. Apparently, he considers these irrelevant in this respect (we shall briefly return to this point below).

Evidently, Kant assumes that the logical structure attributed to a proposition is that of a categorical proposition. Specifically, Kant defines analyticity and syntheticity in terms of certain relations that hold between the subject concept and the predicate concept of a categorical proposition. With Kant, let us henceforth restrict ourselves to affirmative categorical propositions.

In the first *Critique*, Kant provides two examples in order to illustrate the difference between analytic and synthetic propositions. As an example of an analytic proposition Kant gives *all bodies are extended*. This proposition involves the subject concept `body` and the predicate concept `extended`. Kant holds that the concept `extended` is already "contained in" the concept `body` (cf. A7/B11). Hence the proposition is analytic.

Kant suggests that this does not hold for the proposition *all bodies are heavy*. This time, Kant says, "the predicate [i.e., the concept `heavy`] is something entirely different from that which I think in the mere concept of a body in general" (A7/B11). Hence, the proposition is not analytic; in Kant's view, it is to be classified as synthetic.

In passing, we note that Kant would classify theorem 1 at the beginning of § 1.1 also as synthetic (and a priori). The reason is that it is a mathematical theorem.[175] For the sake of clarity, let us state this theorem again:

The sum of the internal angles of every triangle equals two right angles.

Kant would attribute to this theorem the logical structure of a categorical proposition. Reasonably enough, as the subject concept involved in this theorem,

---

[175] The Passage, where Kant describes how he thinks a mathematician proves this theorem (see § 1.1), comes from a section in the first *Critique* where Kant contrasts the philosophical method with the method of pure mathematics; cf. A712/B740.

he would reckon the concept `triangle`. As the predicate concept he would reckon the concept corresponding to the predicate *is such that the sum of the internal angles is equal to two right angles*. We may refer to this concept as `the sum of the (three) internal angles being equal to two right angles` (cf. § 3.1). For the sake of brevity, we refer to this concept as `angles_sum`. Why this is a plausible choice will be argued in § 5.2.

As in the first *Critique*, in the *Prolegomena*, Kant too offers an example of an analytic proposition and an example of a synthetic proposition in order to illustrate the analytic-synthetic distinction (cf. Kant [95], p.62). As an example of an analytic proposition he gives: *all bodies are extended*. Note that this is example is the same as the one given in the first *Critique* (cf. above). As an example of a synthetic theorem, he gives: *some bodies are heavy*. The careful reader will have noticed that this example is not identical to the corresponding example given in the first *Critique*. For the latter reads: *all bodies are heavy*. In the *Prolegomena*, Kant has replaced the quantifier *all* by the quantifier *some*.

Kant's seemingly carefree switch from *all* to *some* suggests that quantifiers (i.e., the respective quantifiers *all* and *some*) are not relevant when it comes to defining what synthetic propositions are. What a synthetic proposition is only bears on the relation between the predicate concept and the subject concept involved in that proposition.

Very likely, a similar point could be made on behalf of analytic propositions. (As we have not found any textual evidence for this, our remarks will remain somewhat speculative in this respect.) Rather than *all bodies are extended*, Kant might as well have taken *some bodies are extended* as an illustrative example of an analytic proposition. The latter proposition would be analytic precisely for the same reason: this time, too, the predicate concept `extended` is contained in the subject concept `body`. As in the case of synthetic propositions, what an analytic proposition is would not turn on quantifiers (i.e., either the quantifier *all* or *some*). Only on the relation between predicate concept and subject concept is relevant.

Now is the time to consider Kant's definitions of analyticity and syntheticity somewhat more closely. We first rephrase Kant's definition of analyticity as in the citation given at the beginning of this section. Let ϕ be a proposition involving the subject concept `A` and the predicate concept `B`. Ignoring quantifiers, we may write ϕ as `A` *is* `B` (cf. also § 3.1).

DEFINITION 1. ϕ is analytic iff the predicate concept `B` is contained in the subject concept `A`.

This definition states an essential characteristic of analytic propositions. As such, it can be taken as a *real definition*.

As to Kant's definition of analyticity, Quine [130], p.21, has once lamented that Kant's talk of a containment relation is left at a metaphorical level. This may indeed seem true for the passage quoted (including the other passages referred to at the end of the quotation). However, Quine's remark puts Kant in a too negative perspective, especially when he means to suggest that there is no elaborate view supporting it. For de Jong [86] has made clear that Kant accepted a view according to which concepts are certain structured entities. Using mathematical terminology, we may interpret de Jong's point as follows: according to Kant, concepts admit at least of a binary, idempotent, commutative and associative "meet operator." This gives concepts an interesting algebraic structure, namely, that of a semi-lattice. Let us briefly explain the point.

For Kant, a concept generally is a "package" or "combination" of other concepts. This can be made articulate by thinking of concepts in algebraic terms. Let $\sqcap$ denote the aforementioned meet operator. As an example, suppose we are given the concepts `animal`, `rational` and `mortal`. Then we can form the combined concept

$$\mathtt{animal} \sqcap \mathtt{rational} \sqcap \mathtt{mortal}.$$ [176]

According to a traditional definition, the latter is also known as the concept `man`. Thus, the operator $\sqcap$ can be seen as one that "combines" concepts in order to form other concepts.

The algebraic structure of concepts gives rise to a containment relation between concepts, as follows: $\mathtt{B}$ is contained in $\mathtt{A}$ iff $\mathtt{A} \sqcap \mathtt{B} = \mathtt{A}$.[177] As a result, definition 1 is equivalent to the following:

DEFINITION 1*. ϕ is analytic iff $\mathtt{A} \sqcap \mathtt{B} = \mathtt{A}$.

For example, since

$$\mathtt{man} \sqcap \mathtt{animal} = \mathtt{man},$$

the concept `animal` is contained in the concept `man`. Hence, the proposition `man` *is* `animal` is analytic.

How does Kant define synthetic propositions? As before, let ϕ be the proposition $\mathtt{A}$ *is* $\mathtt{B}$. Considering the quotation given at the beginning of this subsection, Kant appears to define the syntheticity of ϕ as follows:

DEFINITION 2. ϕ is synthetic iff $\mathtt{B}$ lies entirely outside $\mathtt{A}$.

---

[176] Since $\sqcap$ is clearly associative, we are justified to omit parentheses.
[177] The containment relation is accordingly reflexive, anti-symmetric and transitive, turning it into a partial order.

A qualification should be added to this definition, however. For though the predicate `B` concept lies entirely outside the subject concept `A`, `B` still stands *in a certain connection* with `A`. However, in case of synthetic propositions, this connection is not one—indeed, cannot be one (see below)—of containment.

From definitions 1 and 2, we reasonably enough infer the following:

(α) if a proposition is synthetic, then it is not analytic.

Indeed, if a predicate concept `B` lies entirely outside a subject concept `A`, then we would expect that `B` be not contained in `A`. We shall need this on a later occasion.[178]

The containment relation in case of analytic propositions can be typified as an intensional relation, i.e., it is a relation between *intensions* or *contents*. It is not a relation between *extensions* (of concepts). The extension of a concept is the set of objects falling under that concept. For example, the extension of the concept `human` is the set of humans.[179] Now, if a concept `B` is contained in a concept `A`, then we would expect that the extension of `A` is a subset of the extension of `B`. The following example confirms this: if the concept `animal` is contained in the concept `man`, then the set of humans is a subset of the set of animals.

We think that the connection between the subject concept and the predicate concept in case of synthetic propositions a priori in mathematics is in some way also a "contentual" relation. What would the nature of this relation be? We think that answer must lie in Kant's notion of construction. Recall that for Kant a mathematician always and essentially constructs his concepts, by exhibiting them in terms of an intuition a priori. Accordingly, we may suspect that the content of a concept reappears *in concreto*, as the content of an intuition a priori (this content is provided by an exercise of the productive imagination). We may say that carrying out further constructive procedures provides a "contentual" relation with other intuitions. These further intuitions Kant again sees as concepts *in concreto*. Accordingly, the relation between two concepts in case of a synthetic proposition a priori appears to be a constructive relation. In case of a synthetic proposition, the content of the subject concept is related to the content of the

---

[178] At first sight, it is not entirely clear whether the reverse implication also holds, i.e., that a proposition is synthetic provided it is not analytic. If the reverse implication does hold, then every proposition would in Kant's view be either analytic or synthetic. Hence, in Kant's view, there would be no propositions that are neither analytic nor synthetic.

[179] This is obviously problematic as it stands, since we typically tend to think of a set as a kind of stable thing. However, the number of humans typically varies over time. *Prima facie*, then, there doesn't seem to be a thing such as "the set of humans." The difficulty may be repaired by defining the extension of `man` as the set of *possible* humans. Let us also add that Kant has somewhat different views on extensions of concepts (Kant defines the extension of a concept is the set of concepts that are contained in it); cf. De Jong [86] for a detailed discussion.

predicate concept in that the former can, as it where, be constructively developed into the latter. This "content development," however, always takes place *in concreto*. Indeed, in Kant's view, it is essential for a mathematician to construct his concepts (§ 3.3). This means that a mathematician does not reason conceptually.

Something along these lines seems to be precisely what happens in case of the proof described in The Passage. According to Kant, the mathematician begins with the construction of a triangle. Thus, he exhibits the concept `triangle` *in concreto*, in terms of an intuition a priori. Subsequently he executes further constructive procedures to produce further intuitions and thus to provide a contentual link with the predicate concept `the sum of the (three) internal angles being equal to two right angles` (cf. above).

What we see is that, for Kant, the content of a synthetic a priori proposition in mathematics (a mathematical theorem) does not stand apart from the way it is proved. In this respect, the content and justification of a proposition are closely related.[180] Of course Kant did not speak of theorems and propositions but of *judgments* instead. From a procedural point of view, however, a judgment is a procedure of getting to know that a proposition is true, and thus to affirm the truth of that proposition. Given this, our point can be rephrased as follows: to judge a proposition to be true comes close to proving it. Consequently, the passage may be taken as providing an elaborate description of a synthetic judgment a priori, or, more accurately, a procedure of "synthetically judging a priori."

The principle of analytic propositions. We have seen that Kant defined analytic propositions as those propositions such that the predicate concept is contained in the subject concept. The following citation may lead think that Kant offers yet a second definition of analytic propositions:

> […] *if the judgment is analytic,* whether it be negative or affirmative, its truth must always be able to be cognized sufficiently in accordance with the principle of contradiction (A151/B190; cf. Kant [95], p.62).

That Kant would have given two definitions of analyticity was claimed by Ayer [1], p.104. Upon closer inspection, however, a different reading suggests itself. Let us begin by noting that Kant here states a necessary condition for the *knowability* of (the truth of) an analytic theorem: if a proposition is analytic, then

[180] Compare Frege, who said that the "distinctions between […] synthetic and analytic concern, as I see it, not the content of the judgment but the justification for making the judgment." Remarkably, Frege added a footnote in which he said the following: "By this I do not, of course, mean to assign a new sense to these terms, but only to state more accurately what earlier writers, Kant in particular, have meant by them" (Frege [45], p.3; *ibid*., n.1).

an agent can come to know (i.e., prove it) it solely by means of the principle of contradiction.

Kant calls the principle of contradiction the "proposition that no thing can have a predicate that contradicts it" (A151/B190). Alternatively stated, given two concepts A and B, the principle of contradiction is the principle that no A can be both B and not B.

Let us wonder whether, in Kant's view, the aforementioned condition is also sufficient. The following quotation makes clear that it is:

> […] synthetic judgments … agree in this, that they can by no means arise from […] the principle […] of contradiction; they demand yet a completely different principle (Kant [95], p.62; cf. A151-2/B191; cf. A154/B193).

What Kant appears to say is that if a proposition is synthetic, then one cannot come to know it by means of the principle of contradiction alone. Since synthetic propositions are non-analytic (see (α) above), we can conclude that the principle of contradiction is not only a necessary but also sufficient means for knowing the truth of analytic propositions. Taken together, we have

(β) a proposition is analytic iff its truth can be known in accordance with the principle of contradiction alone.

In other words, a proposition is analytic iff it can be proved in accordance with the principle. Kant explains:

> For the contrary of that which as a concept already lies and is thought in the cognition of the object is always correctly denied, while the concept itself must necessarily be affirmed of it, since its opposite would contradict the object (A151/B190-1).

What is the relation between (β) and definition 1 from the previous subsection? Recall we have taken definition 1 as a real definition of analyticity.

In Kant's view, the principle of contradiction forms a means to an end, namely a means to the end of getting to know the truth of analytic propositions. Accordingly, the principle of contradiction provides a sufficient criterion for getting to know the truth of analytic propositions. This point can be alternatively put as follows.

A *criterion* is a principle or means for deciding (i.e., getting to know) whether entities of a certain kind are in the possession of a certain property (cf. Carney and Scheer [27], p.73, n.3). Notice, however, that a criterion is precisely what is provided by (β): given an analytic proposition, the latter gives a means by way of which it can be decided that it is true.

We conclude that Kant does not give two definitions of analyticity. In contrast, he gives a definition and a criterion instead. The former states what analytic propositions are. The latter states a principle or means by means of which their truth can be decided, i.e., a principle by means of which one can come to know their truth.

Kant considers the principle of contradiction the "supreme principle" of all analytic propositions (cf. A150/B189; cf. also A151/B191). Furthermore, Kant classifies the principle of contradiction as a "merely logical principle" (A153/B192). In fact, Kant acknowledges the principle of contradiction as the highest principle of general logic (cf. A151/B190). It seems to follow that an analytic proposition can be known exclusively by means of general logic. We shall turn to Kant's characterization of general logic in § 4.3.

Besides the principle of contradiction, Kant sometimes gives the impression that analytic propositions can be proved also by means of a different principle:

> Analytic judgments (affirmative ones) are […] those in which the connection of the predicate is thought through identity (A7/B10).

In the same vein, Kant says in the *Jäsche* logic:

> Propositions whose certainty rests on *identity* of concepts (of the predicate with the notion of the subject) are called *analytic* propositions. Propositions whose truth is not grounded on identity of concepts must be called *synthetic* (Kant [93], p.606).

The principle appealed to here is the principle of identity (cf. also Kant [92], p.67; Kant [91], pp.104-5).[181] In case of an analytic proposition `A` *is* `B` (say), the predicate concept `B` is identical with some concept contained in `A` (cf. *ibid*.). Note that accordingly it seems more appropriate to speak of the principle of partial identity. The idea is to think of the subject concept `A` as a "package" or combination of other concepts, namely, the concepts contained in it. The analytic proposition `A` *is* `B` rests on thinking the identity of the predicate concept `B` with some concept contained in this package. We will return to the principle of identity in § 5.2. We now address the following question: what is the criterion for getting to know the truth of synthetic propositions, and the synthetic propositions a priori in mathematics in particular?

The principle of synthetic propositions. Is it possible to come to know the truth of a synthetic proposition solely by means of the principle of contradiction (that is, solely by means of general logic)? Kant's answer to this question is negative:

[181] In the pre-critical work "A new elucidation of the first principles of metaphysical cognition," Kant calls this principle of identity the principle that "*whatever is, is, and whatever is not, is not*" (Kant [90], p.7).

> […] we must allow the *principle of contradiction* to count as the universal and completely sufficient *principle of all analytic cognition*; but its authority and usefulness does not extend beyond this, as a sufficient criterion for truth. For that no cognition can be opposed to it without annihilating itself certainly makes this principle into a *conditio sine qua non*, but not into a determining ground of the truth of our cognition. Since we now [i.e., in the *Critique of pure reason*] really have to do only with the synthetic part of our cognition, we will, to be sure, always be careful not to act contrary to this inviolable principle, but we cannot expect any advice from it in regard to the truth of this sort of cognition (A151-2/B191).

In Kant's view, no item of knowledge (cognition) may violate the principle of contradiction. Hence, no synthetic proposition may violate the principle of contradiction. However, Kant says that we can never expect to come to know the truth of a synthetic proposition by means of the principle of contradiction alone. The principle of synthetic propositions must lie somewhere else. Since for Kant the principle of contradiction is the highest principle of general logic, this seems to mean that the methodological basis for synthetic propositions must lie outside general logic. This is strongly confirmed by the following:

> The explanation of the possibility of synthetic judgments is a problem with which general logic has nothing to do, indeed whose name it need not even know (A154/B193). [182]

Turning to synthetic propositions, and synthetic propositions a priori in particular, Kant holds that the explanation of their possibility is one of the main tasks of transcendental logic:

> But in a transcendental logic it [i.e., the explanation of the possibility of synthetic propositions] is the most important business of all, and indeed the only business, if the issue is the possibility of synthetic *a priori* judgments […] (A154/B193).

For Kant, transcendental logic

> […] has to do merely with the laws of the understanding and reason, but solely insofar as they are related to objects *a priori* […] (A57/B81-2).

In this respect, transcendental logic differs from general logic, which relates to objects a priori as well as a posteriori (cf. *ibid*.).

[182] See also § 5.3.

The principle for getting to know synthetic propositions lies entirely outside general logic. Kant compactly formulates the "supreme principle" of synthetic propositions as follows:

> Every object stands under the necessary conditions of the synthetic unity of the manifold of intuition in a possible experience (A158/B197).

Kant relates the principle for synthetic propositions to intuition. We can now make a plausible guess as to what the principle for synthetic propositions a priori in mathematics will be: a proposition is synthetic a priori, iff its truth can be established by means of constructing concepts in intuition a priori.

Analytic and synthetic propositions differ in the way their constituting concepts are related (see above). However, this section has revealed yet another difference between analytic and synthetic propositions, which is far more important. This time, their difference concerns their principle. In Kant's views, we can come to know the truth of analytic propositions by means of the principle of contradiction alone. In contrast, we can come to know the truth of mathematical theorems by constructing concepts in intuition.

Concluding remarks. In this section, we have shown that Kant defines analytic propositions as those whose predicate concept is contained in the subject concept. A synthetic proposition in contrast, is one whose predicate concept lies entirely outside the subject. Nevertheless, in case of synthetic propositions a priori in mathematics (mathematical theorems) there is still a "contentual link" between the subject concept and the predicate concept. We have suggested that, in case of mathematics, such a contentual link is provided when a mathematician proves a proposition, which he does *in concreto*.

Kant also provides criteria for analytic and synthetic propositions respectively. One can come to know the truth of an analytic proposition solely by means of the principle of contradiction. Since the latter is the highest principle of general logic, the truth analytic propositions can be known by means of general logic alone. Synthetic propositions, in contrast, need an entirely different methodological basis, one that lies outside general logic. However, it should be added that, in Kant's view, no item of knowledge—and hence no mathematical theorem—is allowed to violate the principle of contradiction.

In the case of mathematics, the truth of a proposition can be established on the basis of the possibility of constructing concepts in intuition a priori. The latter, Kant considers a principle of what he calls transcendental logic. Transcendental logic is a logic that refers only a priori to objects. In this respect, transcendental logic differs from general logic, which refers to objects a priori as well as a posteriori.

§ 5.2. How to prove analytic and synthetic propositions: a comparison

In § 4.2, we have seen how, in Kant's view, a synthetic proposition (a theorem) is proved. The aim of this section is to make a comparison between the respective ways analytic propositions and synthetic propositions a priori in mathematics are proved. We begin by investigating Kant's views on the methodology of proving analytic propositions. A comparison with the methodology of proving synthetic propositions follows subsequently.

Proof in action: analytic propositions. We seek a reasonable informative answer to the following question: how, in Kant's view, does one prove an analytic proposition? In order to answer this question, we begin by noting that Kant considers analytic propositions *elucidatory* (A7/B11). Thus, when he speaks of analytic theorems, Kant sometimes parenthetically adds that the predicate concept is "covertly" contained in the subject concept:

> One could call [analytic propositions] *judgments of clarification* […] since through the predicate [they] do not add anything to the concept of the subject, but only break it up by means of analysis into its component concepts, which where already been thought in it (though confusedly) (A7/B11).

In the light of observations like these, de Jong has argued that, for Kant, analytic propositions mainly serve the purpose of conceptual clarification (de Jong [87], p.257). This seems to imply the following: proving an analytic proposition can be characterized as a procedure of conceptual clarification, or *analysis*. Let us explain.

Consider an analytic proposition `A` *is* `B` (say). The predicate concept `B` is contained in the subject concept `A`. To prove the proposition `A` *is* `B` means to "uncover" or to "analytically unpack" the predicate concept `B` from the subject concept `A`. Put differently, to prove `A` *is* `B` means to *analyze* the concept `B` from the concept `A`. Thus, it becomes understandable why Kant would call the analytic theorem `A` *is* `B` elucidatory: the analytic proposition `A` *is* `B` elucidates the subject concept `A`.

Observe that a process of conceptual clarification always starts with a *given* concept (cf. De Jong [87], p.247; cf. A730/B758). Starting from this given concept, another concept is uncovered from it (see below). We can now begin to understand why conceptual clarification can never properly extend our knowledge: in a sense, we only make explicit the knowledge we already have, though perhaps confusedly. Following this line of thought, it becomes tempting to say that in order to expand our knowledge—to go beyond what one already knows—, a certain amount of genuine creativity must be operative. In the case of mathematics, of course, this creativity turns on the workings of the productive imagination (§ 3.4). The latter makes it possible to go beyond the knowledge one already has and thus to obtain new knowledge (see also below). One goes beyond a concept by constructing it, and to provide further constructions subsequently.[183] Let us look in some more detail how the clarification of a concept is generally carried out.

Consider our analytic proposition $\mathtt{A}$ *is* $\mathtt{B}$. In order to prove this proposition, one begins with the concept $\mathtt{A}$, which is considered given, though in a confused or not yet analyzed way. We may imagine that the proof of $\mathtt{A}$ *is* $\mathtt{B}$ proceeds as follows. One begins by analytically unpacking from $\mathtt{A}$ a concept $\mathtt{C}_1$ (say) that is contained in $\mathtt{A}$. Analytically unpacking $\mathtt{C}_1$ from $\mathtt{A}$ can be understood as the execution of what we may refer to as an *analytic procedure*. Let us refer to this analytic procedure as α_1. (A run of an analytic procedure may be called an *analytic process*.) We may say that, given $\mathtt{A}$ as input, a run of α_1 delivers $\mathtt{C}_1$ as its product. We write this as $\mathtt{A}[\alpha_1]\mathtt{C}_1$.

Subsequently, one unpacks from $\mathtt{C}_1$ yet a further concept $\mathtt{C}_2$ that is contained in $\mathtt{C}_1$. Again, this can be understood as the execution of another analytic procedure, which we refer to as α_2. Given $\mathtt{C}_1$ as input, a run of α_2 delivers the concept $\mathtt{C}_2$ as its product. As above, we write this as $\mathtt{C}_1[\alpha_2]\mathtt{C}_2$. In such a way, one proceeds until one arrives at a certain concept $\mathtt{C}_n$. The latter is analytically unpacked from a concept $\mathtt{C}_{n-1}$ by running an analytic procedure we refer to as α_n. From $\mathtt{C}_n$, finally, one unpacks the concept $\mathtt{B}$, which is contained in $\mathtt{C}_n$ by running an analytic procedure we refer to as α_{n+1}. We have: $\mathtt{C}_n[\alpha_{n+1}]\mathtt{B}$.

Collectively, the entire procedure of analytically unpacking $\mathtt{B}$ from $\mathtt{A}$ is formed by sequencing the procedures $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$. This results in a composite analytic procedure that may be written as $\alpha_1;\alpha_2;\ldots;\alpha_{n+1}$ (the same notation for the sequencing of procedures was used in chapter 4). We have:

$$\mathtt{A}[\alpha_1;\alpha_2;\ldots;\alpha_{n+1}]\mathtt{B},$$

which we can alternatively depict:

$$\mathtt{A} \xrightarrow{\alpha_1} \mathtt{C}_1 \xrightarrow{\alpha_2} \mathtt{C}_2 \longrightarrow \;\ldots\ldots\; \xrightarrow{\alpha_n} \mathtt{C}_n \xrightarrow{\alpha_{n+1}} \mathtt{B}$$

[183] The latter belong to what Proclus calls the construction (§ 4.1).

In the light of this, it becomes suggestive to call A the *starting concept* of the analytic procedure depicted above; B may be called the *end concept*. The concepts $C_1, \ldots, C_n$ may be called the *intermediate concepts* of the analytic procedure.

It seems not easy to see how the principle of contradiction plays its role in the above process of conceptual analysis. In contrast, however, it seems considerable less hard to understand how the principle of (partial) identity plays its role in this procedure (see § 5.1). Thus, one executes α_1 precisely because one thinks C_1 to be identical with a certain concept C_1' (say) contained in A. Similarly for all the other analytic procedures α_i ($i \geq 2$). In the light of this, recall that Kant holds that the truth of an analytic proposition can be established by means of the principle of contradiction alone (§ 4.1). However, the impression arising is that in the process of proving an analytic the principle of (partial) identity plays a much more important part—indeed a main part. Perhaps we may say that the principle of contradiction *justifies* our knowledge of (the truth of) an analytic proposition. An analytic procedure of proving an analytic proposition proceeds *in accordance with* the principle of identity. As such, the principle of identity plays its role in the process of proving.

Let us note that in principle every concept can form the starting concept of an analytic procedure, no matter where this concept comes from. For example, Kant suggests that one can analyze the concept triangle. A process of analysis may then deliver, for example, the analytic proposition triangle *is* three_sided as its product (cf. A718-9/B746-7). Remarkable enough, however, Kant considers the proposition gold *is* yellow as an analytic proposition (cf. Kant [95], p.267).[184] This clearly suggests the following. Given the concept gold as the starting point of a process of analysis, the proposition gold *is* yellow may be delivered as the product delivered by this process.

Both the propositions triangle *is* three_sided and gold *is* yellow are in Kant's view analytic and hence a priori. Nevertheless, the concept triangle is a priori while the concept gold is empirical. More generally, then, suppose an item of knowledge C (a concept) is given, although perhaps confusedly. Then in Kant's view, any proposition obtained by analyzing the starting concept C is an analytic proposition (and hence a priori), no matter whether C is a priori or empirical. This leads to a further point.

Let ϕ be an analytic proposition. Let π be an analytic process delivering ϕ as its product. Furthermore, let C be the starting concept of the process π. In Kant's view, the analyticity of ϕ is something independent of the question whether C is a priori or empirical. Though ϕ is analytic (and hence a priori), C may be either a posteriori or a priori. A similar point obviously holds for any of the intermediate concepts figuring in π. In other words, the analyticity of ϕ does

[184] Kant's example is "gold is a yellow metal."

not turn on the respective sources of the items of knowledge figuring in π. In contrast, the analyticity of ϕ seems to turn primarily on the nature of the procedure by means of which the truth of ϕ is established. Specifically, what is important is that this procedure is carried out exclusively in accordance with the principle of (partial) identity, and hence in accordance with general logic alone. An analytic procedure can be easily reconstructed as a series of syllogisms.

A run of the procedure α_1 can be seen as a process of (immediately) getting to know the proposition that $\mathtt{A}$ *is* $\mathtt{C}_1$. Furthermore, a subsequent run of α_2 can be seen as a process of (immediately) getting to know the proposition $\mathtt{C}_1$ *is* $\mathtt{C}_2$. A run of the composite procedure $\alpha_1;\alpha_2$ can be seen as a process of (mediately[185]) getting to know the proposition $\mathtt{A}$ *is* $\mathtt{C}_2$.

We can reconstruct the result of executing the procedure $\alpha_1;\alpha_2$ as a syllogism with middle "term" $\mathtt{C}_2$, as follows:

$$\begin{array}{c} \mathtt{A}\ \textit{is}\ \mathtt{C}_1 \\ \underline{\mathtt{C}_1\ \textit{is}\ \mathtt{C}_2} \\ \mathtt{A}\ \textit{is}\ \mathtt{C}_2. \end{array}$$

In the same vein, we can reconstruct the result of executing the composite procedure $\alpha_1;\alpha_2;\ldots;\alpha_{n+1}$ as a series of syllogisms. This reconstruction proceeds entirely analogous to the way a series of syllogisms was reconstructed from a Lockean proof in § 3.1.

Observe also that, analogous to the case of Lockean proofs, logical notions such as validity and soundness do not straightforwardly apply do an analytic procedure as depicted above. However, they do apply to a syllogistic reconstruction of this analytic procedure. Insofar as the notions of validity and soundness do apply to this procedure, they go firmly hand in hand (cf. § 3.1).

It does not seem that the proof of a synthetic proposition can be reconstructed as a series of syllogisms. We can now clearly see why this is the case. It is precisely because of the auxiliary constructive procedure carried out in Proclus' construction that a mathematical proof does not admit of syllogistic reconstruction. These auxiliary constructions make it that a mathematician does not "unpack" knowledge from given knowledge. In contrast, in Kant's view, a mathematician goes beyond what he knows (see below), and the auxiliary constructions play a pivotal role in this respect.

Cognitive dimensions of analytic and synthetic. Recall that Kant considers analytic propositions of clarification (see above). In contrast with this, synthetic propositions *amplify* our knowledge (this was already suggested above):

[185] The "mediateness" consists precisely in the fact that one first needs to unpack $\mathtt{C}_1$ from $\mathtt{A}$ before $\mathtt{C}_2$ is unpacked from $\mathtt{C}_1$.

> [synthetic propositions could be called] *judgments of amplification*, [since] they do add to the concept of the subject a predicate that was not thought in it at all, and could not have been extracted from it through any analysis (*ibid*.).

This suggests that analytic and synthetic propositions differ in a cognitive sense. Earlier we saw that Kant defined analytic and synthetic propositions in two different ways. Furthermore, they also admitted of two entirely different criteria. However, Kant also holds that analytic and synthetic propositions serve two entirely different cognitive purposes. While the truth of an analytic proposition is established to the end of clarifying our knowledge, synthetic propositions (theorems) are proved to the end of amplifying our knowledge.

For Kant, we saw, the analytic nature of a proposition intimately relates to the nature of the procedure used in order to prove it. An analytic proposition is proved by way of an analytic procedure, one that takes place in accordance with the principle of contradiction. Accordingly, we may broadly typify the method of proving analytic propositions as the *analytic method*. Furthermore, as we saw in chapter 4 (and § 4.2 in particular), a synthetic theorem in mathematics is proved by way of construction. The method of proving mathematical theorems can be typified as *constructive*. It is the method of constructively developing (i.e., expanding) our mathematical knowledge by proving theorems.

Accordingly, a second difference between analytic and synthetic theorems is that their truth is established in two radically different ways. Kant's point is that a mathematician, *qua* mathematician, does not analytically unpack concepts from given concepts. In Kant' view, analysis only makes sense when concepts are given confusedly. In contrast, a mathematician essentially proceeds constructively. The Passage serves to highlight just this essential feature of the mathematical method of proof.

Summary and conclusions. According to Kant, analytic and synthetic propositions differ in several respects.

1. For Kant, all the propositions of mathematics are synthetic and a priori. This holds for the theorem of mathematics in particular. Kant defines analytic and synthetic propositions differently. On the one hand, and analytic proposition is one whose predicate concept is contained in the subject concept. A synthetic proposition, on the other hand, is one whose predicate concept lies entirely outside the subject concept.
2. Kant also provides different criteria for analytic and synthetic propositions. Our knowledge of the truth of an analytic proposition is known in accordance with the principle of contradiction, which Kant considers the highest principle of general logic. However, when one actually proves an analytic proposition, another though related principle plays its part, namely, the principle of (partial) identity. Our knowledge of the truth of a synthetic proposition in mathematics is

based on the possibility of construction concepts in terms of intuitions a priori, which is a principle of transcendental logic.

3. For Kant, analytic and synthetic propositions have a different cognitive purpose. On the one hand, analytic propositions serve to clarify the knowledge we already have, although confusedly. Synthetic propositions, on the other hand, serve to expand our knowledge.

4. Finally, in Kant's view, analytic and synthetic propositions are proved in two radically different ways. An analytic proposition is proved by way of the analytic method. A synthetic theorem in mathematics is proved by way of the constructive method.

§ 5.3. Logic and the mathematical method of proof

In the present section, we discuss Kant's views on the relation between logic and the mathematical method of proof. The issue is important for it considerably improves our understanding as to why Kant claimed that the theorems of mathematics are synthetic, and why, in Kant's view, a mathematical theorem cannot proved by means of general logic alone.

We begin by reviewing some relevant aspects of Kant's reflections on the nature and status of logic. In this respect, we have a keen interest in Kant's distinction between general logic and special logic (this will be explained below). Subsequently, we shall argue that Kant believed that there is what we may call a *special logic of mathematics*, covering at least a method of mathematical proof. We will make a plausible proposal as to what special logic of mathematical proof would look like.

General logic and special logic. In the *Critique of pure reason*, Kant spends several pages on a critical discussion on the nature and status of logic.[186] Kant notes that logic can be undertaken in two different ways. Thus, Kant distinguishes between (1) logic as logic of the general use of the understanding and (2) logic as logic of the special use of the understanding. Says Kant:

> Now logic in turn can be undertaken with two different aims, either as the logic of the general or of the particular use of the understanding. The former contains the absolutely necessary rules of thinking, without which no use of the understanding takes place, and it therefore concerns these rules without regard to the difference of the objects to which it may be directed. The logic of the particular use of the understanding contains the rules for correctly thinking about a certain kind of objects (A52/B76).

[186] What we have in mind are A50/B74-A64/B89, and especially A50/B74-A55/B79.

Instead of the logic of the understanding's general use, Kant sometimes speaks in terms of *general logic*, or *elementary logic* (e.g., A52/B77; A55/B79).[187] We prefer to speak in terms of *general logic*. In Kant's view, general logic concerns the (absolutely necessary) rules of thinking "without regard to the difference of the objects to which it may be directed." The logic of the understanding's special use concerns the rules for thinking correctly about a certain kind of objects (or subject-matter). We shall refer to logic undertaken in this way as *special logic*.

Note that we have used the term *general logic* for several times before in this study (e.g., § 1.3 and § 1.4), putting it more or less on a par with systems of natural deduction. That this is to some extent justified is revealed by the following. One manifest difference between general logic and special logic is that the rules contained by the latter are local in nature while the rules contained by the former are not. In contrast, the rules contained by general logic are topic neutral. In this respect, then, Kant's general logic shares an important characteristic with systems of natural deduction (cf. § 2.1).

Kant sometimes calls a special logic an *organon* (i.e., instrument) of this or that science (A52/B76). Very likely, this must be understood as an instrument to the end of obtaining scientific knowledge in particular. We may expect, then, that an *organon* for a particular science includes a method of proof (of theorems) for that science, and possibly some other things besides. For us, the following question is evidently important: did Kant acknowledge something we may call a *special logic of mathematics*? Such a special logic of mathematics would at least cover a method of proof for mathematics. What would such a method of proof look like? Before propose an answer to this question, let us first take a closer look at general logic.

Kant's general logic is not a logical system as we nowadays think of it (§ 2.1). Regarding Kant's views, *general logic* is best seen as a kind of umbrella term covering such things as the theory of concepts, the theory of judgments, the theory of the syllogism, and the theory of definitions. However, the principle of contradiction and the principle of identity should certainly be mentioned too in this respect (see, e.g., Kant [93]).

As is well-known, Kant took general logic, and syllogistic logic in particular, more or less as he found it. In fact, Kant believed that general logic was already delivered in a more or less completed form by Aristotle (Bviii). Thus, Kant's own contributions to logic merely turn on a few technical refinements of syllogistic logic (e.g., Kant [91]). However, the received opinion is that Kant did not advance the technical content of logic in a very substantial way. Parsons holds that this fact is "striking" and "damaging to his [i.e., Kant's] standing as a philosopher" (Parsons [118], p.115).

[187] Nowadays, the term *elementary logic* is often used to refer what may be broadly called first-order predicate logic.

Parsons remark may suggest that Kant's views on logic are simply uninteresting to consider. However, while it is indeed true that Kant did not contribute very significantly to the technical content of general logic,[188] his philosophical understanding on this score are highly interesting and relevant. Considering the matter in some detail will reveal important aspects of Kant's views on the relation between general logic and mathematical proof.

Kant argues that general logic must not be taken as an instrument (*organon*) to the end of expanding one's knowledge. Kant's words are telling in this respect:

> […] the effrontery of using it [i.e., general logic] as a tool (organon) for an expansion and extension of its information, or at least the pretension of so doing, comes down to nothing but idle chatter, asserting or impeaching whatever one wants with some plausibility (A61-2/B86).

General logic, Kant says, does not—indeed cannot—expand our knowledge. General logic used as an *organon* to the end of expanding our knowledge Kant calls *dialectic*, or a *logic of illusion*. Kant explains that for the ancients

> [dialectic] was the sophistical art for giving to its ignorance, indeed even to its intentional tricks, the air of truth, by imitating the method of thoroughness, which logic prescribes in general, and using its topics for the embellishment of empty pretension. Now one can take it as a certain and useful warning that general logic, *considered as an organon*, is always a logic of illusion, i.e., is dialectical (A61/B86).

For Kant it is wrong to think that by means of general logic alone one can expand one's knowledge. Such is the nature of general logic:

> […] with mere logic no one can venture to make judgments about objects and assert anything about them. Rather, we first must go outside [general] logic to obtain well-based information about objects (A60/B85; cf. also A59-60/B84-5).

A mathematician, *qua* mathematician, typically goes outside general logic, namely, when he constructs his concepts in terms of intuitions a priori (the possibility of constructing concepts accordingly is a principle of transcendental logic). It now becomes very tempting to believe that we need a special logic for mathematics in order to expand mathematical knowledge. Such a special logic for mathematics would contain rules in order to think about a distinctly mathematical subject-matter. We will return to this point shortly.

Meanwhile, let us note that we think that an often-heard complaint against Kant's views on proof only barely scratches the surface of a quite different issue

[188] The most substantial developments began to take place around the turn from the nineteenth to the twentieth century (e.g., Frege [47]).

that may have been much more important for Kant. Thus, it has been frequently thought that Kant had to appeal to intuitions in his account of mathematical proof simply because Kant's general logic was too weak in order to take satisfactory account of mathematical reasoning. Russell expresses the point as follows:

> […] Aristotelian syllogistic theory […] [is] theoretically inadequate to mathematical reasoning, or at any rate required such artificial forms of statement that they could not be practically applied. In this fact lay the strength of the Kantian view, which asserted that mathematical reasoning […] always uses intuitions […]. Thanks to the progress of Symbolic Logic […] this part of the Kantian philosophy is now capable of a final and irrevocable refutation (Russell [137], p.4).

In contrast with what Russell suggests, we think that Kant's views on mathematical proof (according to which mathematical reasoning uses intuitions) did not arise because of the supposed inadequacy of syllogistic logic. For Kant, general logic "analyzes the whole formal business of understanding and reason into its elements" (A60/B84). Accordingly, general logic is formal. In Kant's view, to analyze is precisely what we do when we prove analytic propositions. To this end, general logic, and the principle of contradiction in particular, suffices (§ 5.2).

General logic alone, does not yield us the type of knowledge we are typically after in mathematics. Mathematicians typically want to expand their knowledge, and in Kant's view, one cannot do so by means of general logic alone. What can be achieved by means of general logic is the clarification of one's knowledge, and hence to make explicit the knowledge we already have, though perhaps confusedly. In order to obtain the type of knowledge he is typically after, a mathematician must proceed constructively instead of analytically. Only then he can turn to a definite (i.e., mathematical) content.

As a result, Kant manifests a strong not to take the mathematical method of proof as being the business of general logic. This forms an important difference with a modern, logical conception of proof, which, on the contrary, has a strong tendency to associate logic with methodology (§ 2.4). Considering Kant's views on mathematical proof, an important issue at stake turns on the question: "what is the nature of general logic and what type of knowledge does it yield?" Intimately related to this is the question: "what type of knowledge is mathematical knowledge and how can we get it?"

A special logic for mathematics and Proclus' framework. Our analysis of The Passage in the previous chapter gives us much reason to think that Kant believed that a mathematician proves his theorems in a distinctly mathematical way (cf. § 3.3). This suggests that, in Kant's view, a mathematical proof proceeds by way of a specific and distinct method, which we may refer to as a *mathematical method of proof*. Insights obtained in chapter 3 (and especially § 3.3), show that construction (in the sense of Kant) plays an essential role in this method. In the previous section, we suggested that Kant would have acknowledged a special logic for mathematics. In the present section, we argue that this is indeed the case.

Let us make clear from the start that as far as we can see, we have found no explicit mention of a special logic of mathematics in Kant's work. Therefore, our argument has to be based on other than pure textual considerations. Perhaps it is of some interest first to address the following question: can we mention an example of a special logic in Kant's work?

Tonelli [159], p.81, convincingly argues that the answer to this question must be affirmative. The special logic Kant acknowledged is one that forms the main topic of the *Critique of pure reason*: the transcendental logic.[189] As we have seen, transcendental logic deals with the laws of understanding and of reason, but only insofar as this logic refers a priori to objects. On this point, transcendental logic contrasts with general logic, which refers to objects a priori as well as a posteriori. (See § 5.1.) This makes it plausible that Kant considers transcendental logic as a special logic.

Let us try to provide some idea as to what a special logic for mathematics would look like. As said, a special logic for mathematics is most plausibly understood as an *organon* for thinking about a distinctly mathematical subject-matter (see above). We may expect that such a logic would at least contain a method to the end of proving mathematical theorems.

We propose that Proclus' methodological framework can be considered as in part constituting a special logic of mathematics. More precisely, Proclus methodological framework can be considered as *a special logic of mathematical proofs*. Proclus' framework tells us that a proof of a mathematical theorem (ideally) consists of an enunciation, an exposition, a specification, a construction, an *apodeixis*, and a conclusion. When these items, and the exposition, the construction and the *apodeixis* in particular, are taken in a typically Kantian-constructive sense, then what we get is an instrument to the end of proving theorems in a distinctly mathematical way. Understood in this way, the special logic of mathematics is operative in case of the proof described in The Passage. This much we have seen in § 4.2.

Perhaps the special logic of mathematics can be understood as being in some sense a "restricted" transcendental logic. We have seen that the principle of synthetic propositions a priori in mathematics turns on the possibility of constructing concepts in terms of an intuition a priori. This principle, we saw, is a principle of transcendental logic. However, we may expect that it is also a principle of the special logic of mathematics, and the special logic of mathematical proofs in particular. In other words, then, when we add Proclus'

[189] Ignoring the preface and the introduction, the B edition of the Critique of pure reason counts 884 pages. The Transcendental logic comprises B74-732. A rough estimate then yields that approximately seventy-five percent of the *Critique of pure reason* concerns transcendental logic.

framework to transcendental logic, we may arrive at a special logic for mathematical proofs.

§ 5.4. Conclusion

In this chapter, we have investigated Kant's views on the synthetic a priori in mathematics as well as Kant's views on the relation between logic and the mathematical method of proof. Analytic propositions differ from synthetic propositions in that in case of the former the predicate concept is contained in the subject concept. In the case of synthetic propositions, in contrast, the predicate concept lies outside the subject concept, though it still stands in some connection with the subject concept. Turning to analytic propositions and synthetic propositions in particular, there is a much more important difference, however. Analytic propositions on the one hand and synthetic propositions on the other differ in that they admit of two entirely different principles to the end of proving them. An analytic proposition can be proved solely by means of general logic, and the principle of contradiction in particular. A synthetic proposition, in contrast, requires a completely different methodological principle, namely, the possibility of the construction of concepts in intuition (which is a principle of transcendental logic). As a result, an important difference between analytic and synthetic propositions lies in their method of proof: in Kant's view, analytic and synthetic theorems are proved in two radically different ways. Consequently, analyticity and syntheticity are in part methodological notions. However, analytic and synthetic propositions differ also in another and related way. Analytic propositions serve the cognitive end of clarifying our knowledge, while synthetic propositions serve the cognitive end of expanding it. This points at an important cognitive difference between analytic and synthetic propositions: they constitute different respective types of knowledge. The method of proof in mathematics is constituted by what according to Kant can be properly called a special logic of mathematics, or a special logic of mathematical proofs. Proclus' methodological framework for proving theorems in mathematics can be naturally taken as constitutive for such a special logic.

Concluding remarks

In the present study, we have considered Kant's views regarding the procedure a mathematician carried out in order to prove a theorem. We have concentrated attention on a specific question: what type of item of knowledge does a mathematician employ when he proves a theorem?

Kant recognizes two prime types of item of knowledge: concepts and intuitions. It turns out that intuitions come in two different types: intuitions a posteriori and intuitions a priori. Given an intuition, Kant makes a distinction between form and content. The form of an intuition consists of a collection of relations in space and time. The relata among which these relations hold constitute the content of an intuition. In the present study, we have only considered intuitions insofar as their form consists of a collection of relations in space. (Time needs to be taken into account when one wants to consider, for example, Kant's views on inferences involving continuity. Such is not done in the present work.) This suggests that an intuition is an item of knowledge of a quite specific format. We have proposed to construe an intuition as a diagrammatic item of knowledge.

Now, in Kant's view, a mathematician essentially proves his theorems constructively. This means that a mathematician proves a theorem essentially by way of constructing concepts. To construct a concept (e.g., the concept of a triangle, or a line), in turn, means to exhibit that concept diagrammatically, in terms of an intuition *a priori*. The diagrammatic features of these intuitions a priori, furthermore, are really exploited in the course of a proof. Consequently, in Kant's view, a mathematician employs intuitions a priori in the course of proving a theorem. This provides the answer to the question stated in the first paragraph above.

According to Kant, then, proving a mathematical theorem turns out to be a form of diagrammatic reasoning—indeed it is *essentially* so. Hence, Kant, does not accept a fundamental assumption underlying the logical conception of proof, namely that reasoning is fundamentally reasoning with propositions. As an interesting by-product of our approach, Kant's notion of intuition also comes to stand in an interesting new light: an intuition constitutes a diagrammatic mode of cognitive organization.

On the one hand, for Kant, the form of an intuition (a posteriori or a priori) is given together with the knowing subject; its content, on the other hand, is obtained by exercising certain cognitive faculties or powers. If the content of an intuition is obtained from sense perception (which Kant considers a receptive power), then that intuition is an intuition a posteriori. The content of an intuition a priori in mathematics, in contrast, is obtained by exercising the productive imagination (which Kant considers a spontaneous power). The productive imagination is a power of original exhibition, making it a properly *creative* power. As such, it differs from what Kant calls the reproductive imagination, which, as the term indicates, exhibits by merely reproducing certain elements. These considerations lead to a remarkable conclusion: in Kant's view, the source of (pure) mathematical knowledge lies in the imagination, or, more precisely, the productive imagination.

Kant famously claimed that all the propositions of mathematics—and the theorems in particular—are synthetic and a priori. In Kant's view, synthetic propositions generally (i.e., a priori or a posteriori) amplify our knowledge beyond the knowledge we already have. Analytic propositions, in contrast, merely clarify the knowledge we already have; they make explicit what we already know, though perhaps confusedly. In this respect, the difference between the analytic and the synthetic appears to be primarily a cognitive one: analytic and synthetic propositions constitute different types of (propositional) knowledge.

Kant holds that by means of general logic one is not able to advance knowledge beyond the knowledge we already have. Hence, a mathematician cannot prove theorems by means of general logic alone. General logic does suffice, however, to clarify our knowledge, and hence to get to know analytic propositions. General logic, for Kant, comprises syllogistic logic and the principle of contradiction.

To get to know mathematical theorems is, in Kant's view, accounted for by what we have called a special logic of mathematical proofs. This special logic of mathematical proofs can be understood as distinctly mathematical methodology of (mathematical) proofs, and accordingly concerns the procedure a mathematician carries out when he proves a theorem. What does this methodology look like? We have considered a methodological framework for proving theorems in mathematics due to the neo-platonic philosopher Proclus. We have argued that this framework can be plausibly taken as being to a considerable extent constitutive for Kant's special logic of mathematical proofs.

According to Proclus, one ideally proves a theorem by proceeding through six stages: an enunciation, an exposition, a specification, a construction, an *apodeixis*, and a conclusion. In the enunciation, a theorem is considered. In the exposition, that what is given is taken apart and prepared for use in the rest of the proof. The specification sets forth the goal. This suggests that the enunciation states a certain problem or task. Next, certain auxiliary constructions are carried out in the "construction" and several inferences are made. These inferences concern the *apodeixis*. Finally, in the conclusion, everything is collected and it is confirmed that the theorem stated in the enunciation has been proved.

On the whole, Kant interprets this framework constructively. According to Kant a concept is constructed in the exposition. The product of this construction (an intuition a priori) is further developed by carrying out further constructions after the specification. It turns out that, in Kant's view, the inferences that belong to the *apodeixis* are intimately related to these constructions. So intimately, indeed, that we can consider a mathematical proof as consisting of several constructive-inferential procedures (as we have called them).

Closely related to this is that in Kant's view, a mathematician, *qua* mathematician, proves his theorems in a distinctly mathematical way. For Kant, the mathematical way is not the way of general logic. In particular, one should not confuse general logic with mathematical methodology, and the methodology of mathematical proofs in particular. The mathematical way, we may say in a Kantian spirit, is the way of construction. Without construction, Kant holds, a mathematician cannot make single step, not a single inference, as a means to the end of mathematically establishing the truth of a theorem.

Several questions for future research are still open. Let us raise two points we think are especially interesting.

A first question turns on the functional role of intuitions in the course of a proof. For we have mainly argued *that*, in Kant's view, a mathematician employs intuitions in the course of a proof. However, the question *why*—to what end—a mathematician, *qua* mathematician, would employ precisely this type of item of knowledge has remained largely unanswered. The issue can be developed in at least two directions.

First, some remarks of Kant suggest that he believes that intuitions have a certain heuristic function: they suggest what steps to take in the course of a proof. Related to this, they also prevent a mathematician from executing mistaken steps, by, as Kant says, putting each inference in front of our eyes, so to speak (cf. A734/B762). It would be interesting to come to closer grips with this heuristic aspect of intuitions. Presumably, we will at least need to have a better account of the structural dimensions of intuitions, and of the way the structure of intuitions is employed in the course of a proof.

Second, and to some extent related to the previous point, those who have carried out a mathematical proof by themselves will be prepared to admit that at least some mathematical proofs in some sense make one "see" that a theorem is true, or that a mathematical proof "shows" that a theorem is true (cf. Hardy [63], p.18). In a way, a mathematical proof puts one in an immediate or almost immediate relation with the object of one's knowledge. Something along these lines seems to be suggested by the traditional word *demonstration*—a proof demonstrates a theorem. Kant sometimes appears to believe that this aspect of mathematical proofs is intimately linked to their intuitive character (A735/B763).[190] This seems to be an important aspect of proof. Perhaps Kant's notion of intuition may help one to understand this aspect of mathematical proof.

A second direction for future research lies in a systematic study of Kant's views on proof in algebraically oriented parts of mathematics. Kant's remarks on this point are notoriously brief (A717/B745, A734/B762). Especially Kant's

[190] See also footnote 17.

notion of "symbolic construction" in algebra turns out to be hard to understand (see Shabel [143] for a historical study). Hintikka has suggested interpreting Kant's notion of symbolic construction in terms of modern logic. However, as will perhaps be expected, we think that such a strategy is highly unsatisfactory. Presumably, Kant sees intuitions in algebra as taking the form of concrete algebraic notations, which are manipulated in accordance with certain rules (cf. A717/B745). This suggests that in Kant's view, an item of knowledge can also be formatted as a notation, and not only as a diagram (as in geometry). Consequently, to explore Kant's views on mathematical proof in algebra will require a study of the role of notations in mathematical proof.

Appendix: two proofs from topology

In this study, we have mainly restricted ourselves to proofs from elementary geometry (§ 1.2). In this appendix, we present and briefly discuss two proofs from modern topology. We point out that these proofs can be naturally in terms of Proclus' methodological framework,[191] the respective items of which can be taken in Kantian constructive terms. We accordingly wish to indicate that Kant's views on the mathematical method of proof where not necessarily restricted to the mathematics of his days. The proofs we present have a strong geometrical flavor. The reader is referred to Nelsen [114], [115] for a variety of other geometrical proofs.

Topology. Topology is a branch of mathematics not known in Kant's days. Topology is often roughly characterized as the study of properties of objects—topological spaces—that remain invariant under continuous transformations. The general idea of topology can be understood without delving deep into the mathematics. For definitions and proofs, see Engelking [39].

As an illustrative example, a "ball" can be continuously transformed into, say, a cube: think of the ball as made of rubber; it can be transformed into a cube by kneading and stretching it in appropriate ways.[192] A property of the ball that remains invariant under such kneading and stretching is, for example, that it is made out of one piece (or, as a topologist would say, its "connectedness"): both a ball and a cube are made out of one piece. In a sense, a continuous transformation transfers this property from the ball to the cube.

A continuous transformation of one space to another is sometimes called a *mapping* (e.g., Lefschetz [101], p.33; see also below).

The kneading and stretching of a ball into a cube is invertible: a cube can be continuously transformed back into a ball again. Objects that can be continuously transformed from the one into the other and back again are said to be *homeomorphic*. The ball and the cube are homeomorphic.

An invertible continuous transformation whose corresponding inverse transformation is also continuous is called a *homeomorphism*, or a *topological mapping*. In a sense, a homeomorphism carries over every topological property from one space to another and *vice versa*. Thus, two homeomorphic topological spaces share all their topological properties, and are in that sense "topologically equivalent."

Examples of two spaces that are not homeomorphic are a ball and a torus (i.e., a "donut"). In order to transform a ball into a torus one must "tear" a hole in it at some place. This, however, makes any transformation from a ball to a torus

[191] Thus indicating that even Proclus' methodological framework need not have lost its systematic value.
[192] Hence, topology is sometimes called "rubber sheet geometry."

discontinuous. A topological property of the ball that is not shared by the torus is that the former is simply connected while the latter is not. Roughly, a topological space is called *simply connected* if every closed curve can be shrunk to a point without breaking that curve open.

The first proof. The first example proof we shall consider is from Solomon Lefschetz' *Introduction to topology* [101]. This book was published 1949 and has been reprinted for several times afterwards. We shall not look at Lefschetz proof as a piece of text (broadly understood, so as to include diagrams) but as encoding a proof procedure. By way of preparation, let us present a few definitions and notations.

Where $n \geq 1$, $\mathbb{R}^n$ denotes n-dimensional Euclidean space. A *region* in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$. A region Ω in $\mathbb{R}^n$ is called *bounded* if the distance between every two elements of Ω remains below some fixed bound. Roughly, Ω is called *closed* if it contains all its boundary points.[193] Ω is called convex roughly if for every x, $y \in \Omega$ the line segment from x to y lies entirely within Ω. A *closed n-cell* is any subset of $\mathbb{R}^n$ homeomorphic to the n-dimensional "unit ball" B^n, that is, the set of all points in $\mathbb{R}^n$ whose distance to the origin does not exceed the unit length:

$$B^n = \{(x_1,\dots, x_n) \in \mathbb{R}^n : \Sigma_i x_i^2 \leq 1\}.$$

The theorem Lefschetz proves is:[194]

> (10.2) *A bounded closed convex region* Ω *of* $\mathbb{R}^n$ *is a closed n-cell* (*ibid.*, p.37).

In other words, a bounded closed convex region Ω of $\mathbb{R}^n$ is homeomorphic to B^n.

Lefschetz presents the following proof:[195]

> Let B denote the boundary of Ω and let P be any interior point of Ω ($P \in \Omega - B$) and S a sphere of center P. Denote by H the closed spherical region bounded by S. Any ray issued from P meets B in a single point Q and S in a single point R. Let t be the transformation $\Omega \to H$ defined as follows. If M is any point of the segment PQ then

[193] Consider the set $\{x \in \mathbb{R} : 0 < x < 1\}$. Since this set does not contain its boundary points (i.e., 0 and 1), it is not closed (in $\mathbb{R}$). The set $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$, in contrast, is closed (in $\mathbb{R}$).

[194] The number (10.2) is Lefschetz' original numbering; we have added it only to make understandable Lefschetz' cross reference to the formulation of the theorem at the end of his proof.

[195] The text "Figure 13" underneath the diagram is simply taken over from the original text.

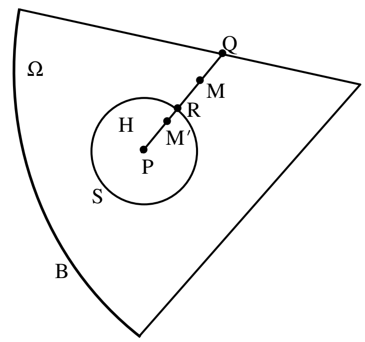


Figure 13

> $M' = tM$ is that point of the segment PR which divides it in the same ratio as M divides PQ. It is easily seen that t is one-one and continuous. Hence since H is Hausdorff and Ω is compact, t is topological (8.4) and since H is a closed n-cell, this proves (10.2) (*ibid*.).

Theorem (8.4) referred to in the final sentence is the following. Let X be a compact space and let Y be a Hausdorff space. If $f : X \to Y$ is a mapping, then $f[X]$ is closed in Y (cf. *ibid*., p.35).

Note that Lefschetz does strictly speaking not prove that Ω is homeomorphic to B^n. In contrast, what he shows is that there is some closed *spherical* region H in $\mathbb{R}^n$ to which Ω is homeomorphic. It is geometrically obvious, however, that any such H is itself homeomorphic to B^n. In order to establish that H is homeomorphic to Ω, a transformation t is defined that "shrinks" Ω in a way such that the result coincides with H. What needs to be shown in order to complete the proof is that t is a homeomorphism (i.e., a topological mapping). To this end, one needs to prove (1) that t is continuous and (2) that t is 1-1 (i.e., invertible). By means of the theorem referred to (i.e., theorem (8.4)), the result follows.

Note that the proof Lefschetz has presented is not carried out in much detail. In particular, Lefschetz does not prove (1) and (2) above. Instead, he merely says that both (1) and (2) are "easily seen," thus leaving the respective proofs to the reader. If only for the sake of argument, we may think of (1) and (2) as two lemmas appealed to in order to prove the theorem Lefschetz has numbered (10.2). However, the details of the respective proofs of (1) and (2) need not concern us here.

We can easily discern the respective items of Proclus' framework in this proof. Thus, Lefschetz' statement of the theorem can be naturally taken as corresponding to the enunciation. In Kantian terminology, the exposition corresponds to the exhibition of a bounded closed convex region of $\mathbb{R}^n$ (which is called Ω). The product of this exhibitive procedure is displayed in the diagram form the quotation above. There appears to be no specification. However, would a specification be given, it would reasonably enough concern the proof of the existence of an homeomorphism $\Omega \to B^n$. The construction concerns the diagrammatic exhibition of:

1. the boundary B of Ω;
2. a point P in the interior of Ω;
3. a sphere S with center P;
4. a closed spherical region H bounded by S;
5. a ray issued from P meeting B in a single point Q and S in a single point R;
6. two points M and M' such that $M'R : PM' = MQ : PM$.

Again, the respective products of 1-6 are displayed in the diagram above. The *apodeixis*, when written out in full detail, would correspond to the proof that the transformation t defined by 6. is one-one and continuous, and that t is "topological" (i.e., a homeomorphism). The conclusion is given at the end where Lefschetz says "this proves (10.2)."

The second proof. The example proof that we shall consider in the current section is taken from Kuratowski's *Introduction to set theory and topology* [99] from 1961.[196]

The theorem that Kuratowski proves is well-known and is stated as follows:

> […] *a square together with its boundary is a continuous image of a segment* […] (*ibid*. [99], p.222).

Kuratowksi refers to this theorem as *Peano's theorem*, named after Peano, who was the first to prove it. What Kuratowski proves is that there exists a (continuous) curve meeting every point of a square. A curve of this type is sometimes called a *space-filling curve* (or a *Peano curve*, named after Peano, who first proved their existence). What is interesting about the proof presented by Kuratowksi is its geometrical character.[197]

---

[196] The book was first published in Polish; an English translation appeared in the same year.

[197] As said earlier, it was indeed Peano [120], who first proved the result stated by Kuratowksi in 1890. A typical trait of Peano's proof was that it proceeded entirely by arithmetic means. This was also observed by Hilbert [68], who in 1891 presented a different proof. Instead of arithmetic methods, Hilbert used "geometrical intuition" (*ibid*., p459). Moore [111] briefly discusses Peano's 1890 paper. Like Hilbert, Moore says that Peano showed that there is a space filling curve "by arithmetic process" (*ibid*., p. 73). Subsequently, Moore briefly discusses Hilbert's geometrical proof. He says that Hilbert made a space-filling curve "luminous to the geometric imagination"

It will turn out that Kuratowksi proves the above theorem for a specific case: the *unit* square is a continuous image of the *unit* interval.[198] However, this easily generalizes to the general case.

As Kuratowski observes, the above theorem is an immediate consequence of a theorem he has proved one page earlier.[199] However, he announces to give yet another proof: "[t]he following is a direct proof of the Peano theorem (given by Sierpiński)."[200]

Kuratowski sometimes writes the unit interval (or segment) [0, 1] as $\mathfrak{I}$; the unit square is written as $\mathfrak{I}^2$. The numbering of the diagrams that occur in the proof is the same as in the original. The theorem ("Theorem 1") referred to near the end of the quotation is the following: the limit of a uniformly convergent sequence of functions is a continuous function (Kuratowski [99], p.139). This is Kuratowksi's proof:[201]

> We divide the square into 9 equal squares and draw in each of them the diagonal as shown in Fig. 11. We divide the segment [0, 1] into 9 equal segments and we transform (linearly) each of them into the corresponding diagonal in the order given in Fig. 11. We denote by f_1 the function thus defined, mapping the segment [0, 1] continuously into the polygonal line consisting of 9 diagonals. We call the squares considered squares of the first approximation.
>
> Next, we divide each of the 9 squares into 9 equal squares; they are the second approximation squares. We

(*ibid.*). In subsequent sections (§§ 7-11) of Moore's paper, we find a "*Geometric determination of Peano's curve* […]." In these sections, Moore presents detailed geometrical proofs of the existence of a Peano curve (among which is Hilbert's and the one presented by Kuratowski). Note that, apparently, the authors mentioned assume a difference between arithmetic methods on the one hand and geometric methods of proof on the other. Furthering the paths taken by the mathematicians just mentioned, Sierpińksi began to study space-filling curves extensively from 1912 onwards. See Sierpiński [146], where among other things, both Peano's and Hilbert's proofs, are discussed. See also Sierpiński [147]. Nowadays, Sierpiński's name is intimately connected with the study of space filling curves. For an elaborate treatment of space filling curves, see Sagan [139].

198 The unit interval (denoted as [0, 1]) is the set is real numbers between 0 and 1, 0 and 1 included; the unit square is the set of pairs of real numbers (x, y) where both x and y range over the unit interval.

199 The theorem is this: every locally arcwise connected continuum ($\neq \emptyset$) is a continuous image of an interval (*ibid*., [99], 221, Theorem 3). The unit square is an example of a locally arcwise connected continuum. Hence, the Peano theorem follows as a special case.

200 Kuratowksi's attribution of the proof to Sierpiński is incorrect. The proof given by Kuratowksi was already given by Moore [111]. See footnote 197. (Sierpiński himself clearly knew of Moore's work; cf., e.g., Sierpiński [146], p.16.)

201 The expressions "Fig 11" and "Fig 12" are taken over from the original.

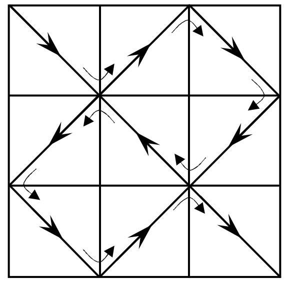


FIG. 11

draw a diagonal D in each of them; here in second approximation squares lying on a diagonal of a first approximation square we draw the diagonal lying on the diagonal D. Thus, the first square of the first approximation appears as in Fig. 11 after the corresponding

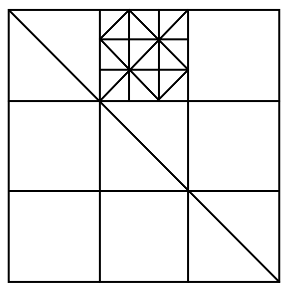


FIG. 12

reduction; the second square of the first approximation is given in Fig. 12.

We divide each of the intervals $(n-1)/9$, $n/9$, where $n = 1, 2, \ldots, 9$, onto 9 equal parts and we map each of these parts onto the square of the second subdivision. This defines the function f_2 which maps the interval [0, 1] continuously onto the polygonal arc made up of 9^2 intervals.

Continuing thus, we define an infinite sequence of continuous functions $f_1, f_2, \ldots, f_n, \ldots$ It is easy to prove that this sequence is uniformly convergent; and therefore its limit function f is continuous (see Chapter XII, § 4, Theorem 1). Furthermore, every point of the square is a value of the function f; in fact, in each square of the n-th approximation there are values of the function f_n and consequently

$$\cup_n f_n(\mathfrak{I}) = \mathfrak{I}^2 \text{ whence } f(\mathfrak{I}) = f(\mathfrak{I}^2).$$

(*ibid.*, [99], pp.222-4).

The strategy of the proof is to define a certain infinite sequence of continuous functions $f_1, f_2, \ldots, f_n, \ldots$ and to consider the limit function f of this sequence.

Each function f_n in the sequence maps the unit interval to the unit square. From the fact that the aforementioned sequence of functions is uniformly convergent (as suggested earlier, this is something Kuratowksi does not show in any detail), it follows that f is a continuous function from the unit interval to the unit square. Finally, it is shown that f is surjective, i.e., is in fact a mapping *onto* the unit square. Thus, f is a space-filling curve as required.

Kuratowski does not present a full proof. In particular, the proof of uniform convergence is left out. The point is not very relevant for us, however. See, for example, Sierpiński [146] for details.

Again, we can easily discern the various items of Proclus' framework. The enunciation corresponds to the theorem stated. When taken in Kantian terms, the exposition concerns the diagrammatic exhibition of the unit square. The product of this exhibitive procedure is displayed in the diagrams. As in the previous case, no specification is given. However, we can easily imagine that a specification, would it be given, concerns the existence of continuous surjection $\mathfrak{I} \to \mathfrak{I}^2$. The construction concerns the diagrammatic exhibition of a sequence of mappings f_1, $f_2, f_3, \ldots$. Note that the sequence $(f_i)_{i \geq 1}$ is produced iteratively. Note also that not all the terms of this sequence are explicitly exhibited in terms of the diagrams occurring in the proof. In fact, the product of exhibiting f_1 is displayed in the first diagram ("fig. 11"). The product of constructing f_2 is only partially in the second diagram (in "fig. 12"). Perhaps the proof turns on the possibility of constructing the remaining terms in the sequence in terms of a certain iterative rule. Subsequently, f is defined as the limit of the sequence. The *apodeixis* concerns the proof that the series $(f_i)_{i \geq 1}$ is uniformly convergent. Hence, f is continuous. Furthermore, it needs to be shown that f is surjective. The conclusion, finally, is not explicitly stated.

References

[1] A.J. Ayer. *Language, truth, and logic*. New York: Dover (1952).

[2] J. Barwise. Heterogeneous reasoning. In: *Conceptual graphs for knowledge representation*, G. Mineau, B. Moulin, J. Sowa (eds.), Berlin: Springer Verlag, Lecture Notes in Artificial Intelligence (1993), 64-74.

[3] J. Barwise, J. Etchemendy. Visual information and valid reasoning. In: *Visualization in teaching and learning mathematics*, W. Zimmerman, S. Cunningham (eds.), Washington: Mathematical Association of America (1991), 9-24. Repr. in: *Logical reasoning with diagrams*, G. Allwein, J. Barwise (eds.), Oxford: Oxford University Press (1996).

[4] J. Barwise. J.Etchemendy: Hyperproof: logical reasoning with diagrams. In: *Reasoning with diagrammatic representations. Papers from the AAAI spring symposium*, technical report no. SS-92-02, N.H. Narayanan (ed.), Menlo Park: AAAI Press (1992), 77-81.

[5] J. Barwise, J. Etchemendy. *Hyperpoof* (software package included). CSLI Publications, Center for the Study of Language and Information: Stanford University, Cambridge: Cambridge University Press (1995).

[6] J. Barwise, J. Etchemendy. Heterogeneous logic. In: Glasgow, Narayanan, Chandrasekaran [58], 211-34.

[7] J. Barwise, J. Etchemendy. Computers, visualization, and the nature of reasoning. In: *The digital phoenix: how computers are changing philosophy*, T.W. Bynum, J.H. Moor (eds.), London: Blackwell (1998), 93-116.

[8] J. Barwise, J. Etchemendy. *Language, proof and logic* (software package included). New York: Seven Bridges Press, Stanford: CSLI Publications, Center for the Study of Language and Information, Stanford University (1999).

[9] J. Barwise, S. Feferman (eds.). *Model-theoretic logics*. New York: Springer-Verlag (1985).

[10] J. Barwise, E. Hammer. Diagrams and the concept of a logical system. In: *What is a logical system?*, D.M. Gabbay (ed.), Oxford: Clarendon Press (1994), 73-106.

[11] N. Belnap. Restricted quantification and conditional assertion. In: *Truth, syntax and modality: proceedings of the Temple University conference on alternative semantics*, H.Leblanc (ed.), Amsterdam: North-Holland (1973), 48-75.

[12] J.F.A.K. van Benthem. Logische theorie en wiskundige praktijk. *Euclides* 55 (1979), 249-54.

[13] J.F.A.K. van Benthem. *Exploring logical dynamics*. Stanford: CSLI Publications, Center for the Study of Language and Information, Stanford University (1996).

[14] J.F.A.K. van Benthem. Is there still logic in Bolzano's key? ILLC Publications, Prepublication (PP) Series (2002). Available at: `http://www.illc.uva.nl/Publications/ResearchReports/PP-2003-03.text.pdf`

[15] P. Bernays. Hilbert's significance for the philosophy of mathematics. In: *From Brouwer to Hilbert. The debate on the foundations of mathematics in the 1920s*, P. Mancosu (ed.), New York: Oxford University Press (1998), 189-97.

[16] E.W. Beth, Über Locke's "allgemeines Dreieck." *Kant-Studien* 48 (1956-57), 361-80.

[17] E.W. Beth. *La crise de la raison et la logique*. Paris: Gauthier-Villars, Collection de logique mathématique, série A, fasc. XII (1957).

[18] E.W. Beth, *Aspects of modern logic*. Dordrecht: Reidel (1970).

[19] D. Bonevac. *Deduction. Introductory symbolic logic*, 2nd ed. Oxford: Blackwell (2003).

[20] G. Boole. *An investigation into the laws of thought*. London: Macmillan (1854).

[21] G. Brittan. *Kant's theory of science*. Princeton: Princeton University Press (1978).

[22] L.E.J. Brouwer. Conscious, philosophy and mathematics. In: *Proceedings of the 10th international congress of philosophy, Amsterdam 1948* (1948), 1235-1249.

[23] L.E.J. Brouwer. *Collected works I. Philosophy and foundations of mathematics*. A. Heyting (ed.), Amsterdam: North-Holland (1975).

[24] S.R. Buss. An introduction to proof theory. In: *Handbook of proof theory*, S.R. Buss (ed.), Amsterdam: Elsevier (1998), 1-78.

[25] S.K. Card, J.D. Mackinlay, B. Schneiderman (eds.). *Readings in information and visualization: using vision to think*. San Francisco: Morgan Kaufmann (1999).

[26] R. Carnap. The logicist foundations of mathematics. In: *Philosophy of mathematics: selected readings*, 2nd ed., H. Putnam, P.Benacerraf (eds.), Cambridge: Cambridge University Press (1983), 41-52.

[27] J.D. Carney, R.K. Scheer. *Fundamentals of logic*. New York: Macmillan (1964).

[28] B. Chandrasekaran. Diagrammatic representation and reasoning: some distinctions. Paper presented at the AAAI Fall 1997 Symposium Series on Diagrammatic Reasoning, Boston (1997). Available at: `http://www.cse.ohio-state.edu/~chandra/dr-ll.pdf`

[29] V. Chappell. Locke's theory of ideas. In: *The Cambridge companion to Locke*, V. Chappell (ed.), Cambridge: Cambridge University Press (1994), 26-55.

[30] C. Chihara. *Constructibility and mathematical existence*. Oxford: Oxford University Press (1991).

[31] I.M. Copi. *Introduction to logic, 4th ed*. New York: MacMillan (1972).

[32] D. van Dalen. *Logic and structure*, 3rd ed. Berlin: Springer Verlag (1997).

[33] M. Detlefsen. Poincaré *vs*. Russell on the rôle of logic in mathematics. *Philosophia mathematica* 3 (1993), 24-49.

[34] M. Detlefsen. Constructivism. In: *Companion encyclopedia of the history and philosophy of the mathematical sciences*, vol. 1, I. Grattan-Guinness (ed.), London: Routledge (1994), 656-64.

[35] M. Detlefsen. Constructive existence claims. In: *The philosophy of mathematics today*, M. Schirn (ed.), Oxford: Clarendon Press, (1998), 307-35.

[36] R. Descartes. *Rules for the direction of the mind*. In: *The philosophical writings of Descartes*, vol. 1, J. Cottingham, R. Stoothoff, D. Murdoch (eds.), Cambridge: Cambridge University Press (1985), 7-78.

[37] M. Dummett. Frege. *Philosophy of language, 2nd ed*. Cambridge: Harvard University Press (1981).

[38] Y. Engelhardt. *The language of graphics*. Ph.D. thesis, Amsterdam: ILLC Dissertation Series, Institute for Logic, Language and Computation, University of Amsterdam (2002).

[39] R. Engelking. *General topology*. Berlin: Heldermann Verlag (1989).

[40] J. Etchemendy. *The concept of logical consequence*. Cambridge: Harvard University Press (1990).

[41] Euclid. *The thirteen books of The Elements*, 2nd ed., 3 vols. Transl., intr. and comm. by T.L. Heath, New York: Dover (1956).

[42] L. Euler. *Letters of Euler on different subjects in natural philosophy: addressed to a German princess,* vol. 1. Transl. and notes by D. Brewster, New York: Arno Press (1975).

[43] D. Fallis. Intentional gaps in mathematical proofs. *Synthese* 134 (2003), 45-69.

[44] K. Fine. *Reasoning with arbitrary objects*. Oxford: Blackwell (1985).

[45] G. Frege. *The foundations of arithmetic: a logico-mathematical enquiry into the concept of number*, 2nd ed. Transl, with German text, by J.L. Austin, Oxford: Blackwell (1953).

[46] G. Frege. *The basic laws of arithmetic: exposition of the system*. Transl., ed. and intr. by M. Furth, Berkeley: The University of California Press (1964).

[47] G. Frege. *Begriffsschrift*, *a formula language, modeled upon that of arithmetic, for pure thought*. In: *From Frege to Gödel. A source book in mathematical logic, 1879-1931*. Cambridge: Harvard University Press (1967), 1-82.

[48] G. Frege. Logic in mathematics. In: *Gottlob Frege: posthumous writings*, H. Hermes, F. Kambartel, F. Kaulbach (eds.), Oxford: Blackwell (1979), 203-50.

[49] M. Friedman. *Kant and the exact sciences*. Cambridge: Harvard University Press (1992).

[50] M. Friedman. Geometry, construction, and intuition in Kant and his successors. In: *Between logic and intuition. Essays in honor of Charles Parsons*, G. Sher, R. Tieszen (ed.), Cambridge: Cambridge University Press (2000), 186-218.

[51] G. Furnas. Reasoning with diagrams only. In: N.H. Narayanan (ed.), *AAAI spring symposium on reasoning with diagrammatic representations: working notes*, Cambridge: AAAI Press, American Association for Artificial Intelligence (1992), 118-23.

[52] D.M. Gabbay. A general theory of structured consequence relations. In: *Substructural logics*, K. Došen, P. Schroeder-Heister (eds.), Oxford: Clarendon Press (1993).

[53] L.T.F. Gamut. *Logic, language, and meaning, volume 2: intensional logic and logical grammar*. Chicago: University of Chicago Press (1991).

[54] H. Gelernter. Realization of a geometry-theorem proving machine. *Proceedings of an international conference on information processing*, Paris: UNESCO House (1959), 271-82. Repr. in *Computers and thought*, E.A. Feigenbaum, J. Feldman (eds.), New York: McGraw-Hill, and Menlo Park: AAAI Press (1963), 134-63.

[55] G. Gentzen. Untersuchungen über das logische Schliessen. *Mathematische Zeitschrift* 39 (1935), 176-221, 405-31.

[56] G. Gentzen. Investigations into logical deduction. *American philosophical quarterly* 1 (1964), 288-306. (Contains a translation of the first part of Gentzen [56]; a translation of the second part appeared in *American philosophical quarterly* 2 (1965), 204-18.)

[57] J. Glasgow, D. Papadias. Computational imagery. *Cognitive science* 16 (1992), 355-94. Reprinted in Glasgow, Narayanan, Chandrasekaran [58], 435-80.

[58] J. Glasgow, N. Narayanan, B. Chandrasekaran (eds.). *Diagrammatic reasoning. Cognitive and computational perspectives.* Cambridge: MIT Press, and Menlo Park: AAAI Press (1995).

[59] C. Glymour. *Thinking things through. An introduction to philosophical issues and achievements*. Cambridge: MIT Press (1992).

[60] M. Greaves. *The philosophical status of diagrams*. Stanford: CSLI Publications, Center for the Study of Language and Information, Stanford University (2002).

[61] H. Hahn. Logic, mathematics and knowledge of nature. In: *Logical postivism*, A.J. Ayer (ed.), New York: The Free Press (1959), 147-61.

[62] E. Hammer. *Logic and visual information*. Stanford: CSLI Publications, Center for the Study of Language and Information, Stanford University (1995).

[63] G.H. Hardy. Mathematical proof. *Mind* 38 (1929), 1-25.

[64] J. Heal. *Mind, reason and imagination*. Cambridge: Cambridge University Press (2003).

[65] J. van Heijenoort. Logic as calculus and logic as language. *Synthese* 17 (1967), 324-30.

[66] C.G. Hempel. On the nature of mathematical truth. In: *Philosophy of mathematics: selected readings*, 2nd ed., H. Putnam, P.Benacerraf (eds.), Cambridge: Cambridge University Press (1983), 377-402.

[67] D.E. Hesseling. *Gnomes in the fog: the reception of Brouwer's intuitionism in the 1920s*. Ph.D. thesis, Utrecht: Universiteit Utrecht, Faculteit Wiskunde en Informatica (1999).

[68] D. Hilbert. Ueber die stetige Abbildung einer Linie auf ein Flächenstück. *Mathematische Annalen* 38 (1891), 459-60.

[69] D. Hilbert. *Grundlagen der Geometrie*. Leipzig: Teubner (1899).

[70] D. Hilbert. Axiomatisches Denken. *Mathematische Annalen* 78 (1918), 405-15.

[71] D. Hilbert. The foundations of mathematics. In: *From Frege to Gödel. A source-book in mathematical logic, 1879-1931*, J. van Heijenoort (ed.), Cambridge: Harvard University Press (1967), 464-79.

[72] K.J.J. Hintikka. Kant's "New Method of Thought" and his theory of mathematics. *Ajatus* 27 (1965), 37-47. Repr. in: *Knowledge and the known: historical perspectives in epistemology*, Dordrecht: Reidel (1974), 126-34. References to the latter.

[73] K.J.J. Hintikka. Kant on the mathematical method. *The monist* 51 (1967), 352-75. Repr. in: *Knowledge and the known: historical perspectives in epistemology*, Dordrecht: Reidel (1974), 160-83. References to the latter.

[74] K.J.J. Hintikka. On Kant's notion of intuition (*Anschauung*). In: *The first Critique: reflections on Kant's Critique of Pure Reason*. T. Penelhum, J. Macintosh (eds.), Belmont: Wadsworth (1969), 38-53.

[75] K.J.J. Hintikka. Kantian intuitions. *Inquiry* 15 (1972), 341-5.

[76] K.J.J. Hintikka. *Logic, language-games and information: Kantian themes in the philosophy of logic*. Oxford: Clarendon Press (1973).

[77] K.J.J. Hintikka. On the logic of an interrogative model of scientific inquiry. *Synthese* 47 (1981), 69-83.

[78] K.J.J. Hintikka. *Lingua universalis vs. calculus ratiocinator: an ultimate presupposition of twentieth-century philosophy*. Dordrecht: Kluwer Academic Publishers (1997).

[79] K.J.J. Hintikka, U. Remes. *The method of analysis. Its geometrical origin and its general significance*. Dordrecht: Reidel (1974).

[80] R.E. Horn. *Visual language. Global communication for the 21st century*. Bainbridge Island: MacroVU, Inc. (1998).

[81] R. Howell. Intuition, synthesis, and individuation in the *Critique of pure reason*. *Noûs* 7 (1973), 207-32.

[82] P. Hoyningen-Heune. Context of discovery and context of justification. *Studies in the history and philosophy of science* 18(4), 501-15 (1987).

[83] S. Isard. Changing the context. In: *Formal semantics of natural language*, E.L. Keenan (ed.), Cambridge: Cambridge University Press (1975), 287-96.

[84] Y. Iwasaki. Problem solving with diagrams: introduction. In: Glasgow, Narayanan, Chandrasekaran [58], 657-67.

[85] M. Jamnik. *Mathematical reasoning with diagrams. From intuition to automation*. Stanford: CSLI Publications, Center for the Study of Language and Information, Stanford University (2001).

[86] W.R. de Jong. Kant's analytic judgments and the traditional theory of concepts. *Journal of the history of philosophy* 33 (1995), 613-41.

[87] W.R. de Jong. How is metaphysics as a science possible? Kant on the distinction between philosophical and mathematical method. *Review of metaphysics* 49 (1995), 235-74.

[88] W.R. de Jong. Kant's theory of geometrical reasoning and the analytic-synthetic distinction. On Hintikka's interpretation of Kant's philosophy of mathematics. *Studies in the history and philosophy of science* 28 (1997), 141-66.

[89] I. Kant. *Anthropology from a pragmatic point of view*. Transl. by M.J. Gregor, The Hague: Martinus Nijhoff (1974).

[90] I. Kant, A new elucidation of the first principles of metaphysical cognition. In: *Theoretical philosophy, 1755-1770*, transl. and ed. by D. Watford, Cambridge: Cambridge University Press (1992), 1-45.

[91] I. Kant. The false subtlety of the four syllogistic figures. In: *Theoretical philosophy, 1755-1770*, transl. and ed. by D. Watford, Cambridge: Cambridge University Press (1992), 84-105.

[92] I. Kant. The Blomberg logic. In : *Lectures on logic*, transl. and ed. by J. M. Young, Cambridge: Cambridge University Press (1992), 1-246.

[93] I. Kant. The Jäsche logic. In : *Lectures on logic*, transl. and ed. by J. M. Young, Cambridge: Cambridge University Press (1992), 517-640.

[94] I. Kant. *Critique of pure reason*. Transl. and ed. by P. Guyer, A. W. Wood, Cambridge: Cambridge University Press (1998).

[95] I. Kant. *Prolegomena to any future metaphysics that will be able to come forward as science*. In: *Theoretical philosophy after 1781*, Transl. and ed. by H. Allison *et al*, Cambridge: Cambridge University Press (2002), 49-169.

[96] I. Kant. *Critique of the power of judgment*. Transl. by P. Guyer, E. Matthews, Cambridge: Cambridge University Press (2002).

[97] P. Kitcher, *The Nature of mathematical knowledge*. Oxford: Oxford University Press (1983).

[98] Z. Kulpa. Diagrammatic representation and reasoning. *Machine GRAPHICS & VISION* 3 (1994), 77-103.

[99] K. Kuratowski. *Introduction to set theory and topology*. Transl. from the rev. Polish edition by L. F. Boron, Oxford: Pergamon Press (1961).

[100] J.H. Larkin, H.A. Simon. Why a diagram is (sometimes) worth ten thousand words. *Cognitive science* 11 (1987), 65-99. Repr. in Glasgow, Narayanan, Chandrasekaran [58], 69-109. References to the latter.

[101] S. Lefschetz. *Introduction to topology*. Princeton: Princeton University Press (1949).

[102] G.W. Leibniz. *New essays on human understanding*. Transl. and ed. by P. Remnant, J. Bennett, Cambridge: Cambridge University Press (1996).

[103] O. Lemon. Review of E. Hammer: Logic and Visual Information. *Journal of logic, language, and information* 6 (1997), 213-6.

[104] O. Lemon, I. Pratt. Spatial logic and the complexity of diagrammatic reasoning. *Machine GRAPHICS & VISION* 6 (1997), 89-108.

[105] R.K. Lindsay. Images and inference. *Cognition* 29 (1988), 229-50. Reprinted in: Glasgow, Narayanan, Chandrasekaran [58], 111-35. References to the latter.

[106] R.K. Lindsay. Using diagrams to understand geometry. *Computational intelligence* 14 (1998), 238-72.

[107] J. Locke. *An essay concerning human understanding*. P.H. Nidditch (ed.), Oxford: Clarendon Press (1979).
[108] J. Lyons. *Introduction to theoretical linguistics*. Cambridge: Cambridge University Press (1968).
[109] B. Mates. *Elementary logic, 2nd ed*. New York: Oxford University Press (1972).
[110] J.D. Monk. *Mathematical logic*. New York: Springer (1976).
[111] E.H. Moore. On certain crinkly curves. *Transactions of the American Mathematical Society* 1(1) (1900), 72-90.
[112] I. Mueller. *Philosophy of mathematics and deductive structure in Euclid's Elements*. Cambridge: MIT Press (1981).
[113] R. Muskens, J. van Benthem, A. Visser. Dynamics. In: *Handbook of logic and language*, J. van Benthem, A. ter Meulen (eds.), Cambridge: MIT Press (1997), 587-648.
[114] R.B. Nelsen. *Proofs without words. Exercises in visual thinking*. Washington: Mathematical Association of America (1993).
[115] R.B. Nelsen. *Proofs without words II. More exercises in visual thinking*. Washington: Mathematical Association of America (2001).
[116] Newell, A., H.A. Simon. *Human problem solving*. Englewood Cliffs, NJ: Prentice Hall (1972).
[117] T. Nickles (ed.). *Scientific discovery, logic, and rationality. Boston studies in the philosophy of science 60*. Dordrecht: Reidel (1980).
[118] C. Parsons. Kant's philosophy of arithmetic. In: *Mathematics and philosophy. Selected essays*. Ithaca: Cornell University Press (1983), 110-49. Reprinted in: *Kant's philosophy of mathematics: modern essays*, C.J. Posy (ed.), Dordrecht: Kluwer (1992), 43-79. References to the latter.
[119] C. Parsons. The transcendental aesthetic. In: *The Cambridge companion to Kant*, P. Guyer (ed.), Cambridge: Cambridge University Press (1992), 62-100.
[120] G. Peano. Sur une courbe, qui remplit toute une aire plane. *Mathematische Annalen* 36 (1890), 157-60.
[121] C.S. Pierce. *The collected papers of C.S. Pierce*. Cambridge: Harvard University Press (1960).
[122] G. Pitcher. *A theory of perception*. Princeton: Princeton University Press (1971).
[123] H. Poincaré. *Science and method*. In: *The foundations of science*, transl. and ed. by G. B. Halsted, Washington: University Press of America (1982), 9-197.
[124] Polya, G. *Mathematics and plausible reasoning*. 2 vol.'s, Princeton: Princeton University Press (1954).
[125] Polya, G. *Mathematical discovery. On understanding, learning and teaching problem solving*. 2 vol.'s, New York: John Wiley and Sons (1962).
[126] K.R. Popper. *The logic of scientific discovery*. London: Hutchingson (1959).
[127] C.J. Posy. Mathematics as a transcendental science. In: *Phenomenology and the logical sciences*, T. Seebohm, D. Føllesdal, J. Mohanty (eds.), Dordrecht: Kluwer (1991), 107-31.

[128] D. Prawitz. *Natural deduction. A proof-theoretical study*. Stockholm: Almqvist & Wiksell (1965).
[129] Proclus. *A commentary on the first book of Euclid's The Elements*. Transl. by G.R. Morrow, Princeton: Princeton University Press (1970).
[130] W.V.O. Quine. Two dogmas of empiricism. In: *From a logical point of view*, 2nd ed., Cambridge: Harvard University Press (1961), 20-46.
[131] S. Ramanujan. *Collected papers of Srinivasa Ramanujan*. G.H. Hardy, P.V. Seshu Aiyar, B.M. Wilson (eds.), Cambridge: Cambridge University Press (1927).
[132] Y. Rav. Why do we prove theorems? *Philosophia mathematica* 7 (1) (1999), 5-41.
[133] H. Reichenbach. *The philosophy of space and time*. Transl. by M. Reichenbach and J. Freund, intr. by R. Carnap, New York: Dover (1958).
[134] H. Reichenbach. *Experience and prediction*. Chicago: Chicago University Press (1961).
[135] R. Rood. Proof, cognition, and rationality. *Logique et analyse* 179-80 (2002), 399-419.
[136] R. Rood. On the status of axioms in mathematics. In: *Basic belief and basic knowledge. Papers in epistemology*, R. van Woudenberg, S. Roeser, R. Rood (eds.), Frankfurt: Ontos Verlag (2005), 79-105.
[137] B. Russell, *The principles of mathematics*, 2nd ed. London: Allen and Unwin (1937).
[138] B. Russell. *Introduction to mathematical philosophy*. London: George Allen and Unwin Ltd. (1919), 145.
[139] H. Sagan. *Space filling curves*. Berlin: Springer-Verlag (1994).
[140] W.C. Salmon. Bayes's theorem and the history of science. *Minnesota studies in the philosophy of science 5: historical and philosophical perspectives of science*, R. H. Stuewer (ed.) Minneapolis: University of Minnesota Press (1970).
[141] W.C. Salmon. *Logic*, 3rd ed. Englewood Cliffs: Prentice-Hall (1985).
[142] P. Scotto di Luzio. Logical systems and logicality. In: *Theory and application of diagrams*, M. Anderson, P. Cheng, V. Haarslev (eds.), Berlin: Springer Verlag, Lecture Notes in Artificial Intelligence (2000), 117-31.
[143] L. Shabel. Kant on the "symbolic construction" of mathematical concepts. *Studies in the history of the philosophy of science* 29 (1998), 589-621.
[144] A. Shimojima. Operational constraints in diagrammatic reasoning. In: *Logical reasoning with diagrams*, J. Barwise, G. Allwein (eds.), Oxford: Oxford University Press (1996).
[145] S.-J. Shin. *The logical status of diagrams*. Cambridge: Cambridge University Press (1994).
[146] W. Sierpiński. O krzywych, wypełŋiających kwadrat (Sur les courbes qui remplissent un carré). *Prace mathematyczno-fizyczne* 23 (1912), 193-219.

[147] W. Sierpiński. Sur une nouvelle courbe continue qui remplit tout une aire plane. *Bullentin international de l'Academie Polonaise des Sciences et des Lettres, Cracovie, Serie A, sciences mathématiques* (1912), 462-78.

[148] A. Sloman. Interactions between philosophy and AI: the role of intuition in non-logical reasoning and intelligence. *Artificial intelligence* 2 (1971), 209-25.

[149] F. Sommers, G. Englebretsen. *An invitation to formal reasoning: the logic of terms*. Aldershot: Ashgate (2000).

[150] J.F. Sowa. *Conceptual structures: information processing in mind and machine*. London: Addison Wesley (1984).

[151] M. Steiner. *Mathematical knowledge*. Ithaca: Cornel University Press (1975).

[152] I. Stewart. *Galois theory, 2nd ed*. London: Chapman & Hall (1989).

[153] G. Sundholm. Questions of proof. *Manuscrito* 16 (1993), 47-70.

[154] G. Sundholm. Semantics for natural deduction derivations. Unpublished lecture (1998).

[155] N. Tennant. The withering away of formal semantics? *Mind and language* 1 (1986), 302-18.

[156] M. Thompson. Singular terms and intuitions in Kant's epistemology. *Review of metaphysics* 26 (1972), 314-43. Repr. with postscript in: *Kant's philosophy of mathematics: modern essays*, C.J. Posy (ed.), Dordrecht: Kluwer (1992), 81-107. References to the latter.

[157] P. Tidman, H. Kahane. *Logic and philosophy*, 9th ed. Belmont: Wadsworth: Thompson (2003).

[158] P. Tomassi. *Logic*. London: Routledge (1999).

[159] G. Tonelli. *Kant's Critique of Pure Reason within the tradition of modern logic*. Zürich: Georg Olms (1994).

[160] J. Venn. *Symbolic logic*. New York: Chelsea (1884).

[161] C.F.M. Vermeulen. Text structure and proof structure. *Journal of logic, language, and information* 9 (2000), 273-311.

[162] D. Winterstein, A. Bundy, M. Jamnik. A proposal for automating diagrammatic reasoning in continuous domains. In: *Theory and application of diagrams*, M. Anderson, P. Cheng, V. Haarslev (eds.), Berlin: Springer-Verlag, Lecture Notes in Artificial Intelligence (2000), 286-99.

[163] L. Wittgenstein. *Tractatus logico-philosophicus*. Transl. by C.K. Ogden, London: Kegan Paul (1933).

Samenvatting

Hoe bewijs je een wiskundig theorema? In dit proefschrift richten we ons op een aspect van Kants opvattingen omtrent de wiskundige bewijsmethode. We beperken ons hierbij tot Kants opvattingen aangaande bewijzen in de elementaire geometrie. We presenteren twee geometrische bewijzen uit de moderne topologie in een appendix (deze bewijzen hebben een sterk geometrisch karakter). Aldus willen we suggereren dat Kants ideeën ook goed toepasbaar zouden kunnen zijn op de wiskunde van na zijn tijd.

Het aspect van Kants bewijsopvatting waarin we geïnteresseerd zijn kan aan de orde gebracht worden door het stellen van een specifieke vraag.

Wanneer iemand redeneert (zoals bijvoorbeeld een wiskundige die een theorema bewijst), dan kunnen we een onderscheid maken tussen datgene *waarmee* deze persoon redeneert en datgene *waarover* deze persoon redeneert. Datgene waarmee geredeneerd wordt kunnen we *kennis* noemen (breed opgevat). Het is deze kennis waar we ons op zullen concentreren.

Nader beschouwd kan de kennis waarmee iemand redeneert opgedeeld worden in verschillende "kennisquanta" of, zoals wij ze noemen, *kennisitems*. Een natuurlijke vraag die nu gesteld kan worden is: wat voor type kennisitems gebruikt een wiskundige wanneer hij een theorema bewijst? We benaderen Kants visie op de methodologie van wiskundig bewijzen in de hoofdzaak met deze vraag in ons achterhoofd. Een groot deel van dit proefschrift (in het bijzonder de hoofdstukken 2-5) kan gezien worden als Kants uitgebreide antwoord op deze vraag. De volgende beknopte argumentatie leidt tot Kants antwoord.

Een wiskundige, *qua* wiskundige, redeneert volgens Kant essentieel op constructieve wijze. Dit betekent dat een wiskundige zijn theorema's bewijst door middel van het construeren van begrippen (bijvoorbeeld: het begrip van een driehoek, of van een lijn). Een begrip construeren betekent: een met dat begrip corresponderende intuïtie (of *Anschauung*, zoals Kant zelf zou zeggen) verbeelden. In Kants visie is een intuïtie ruwweg een kennisitem bestaande uit relaties in ruimte en tijd. Met andere woorden: een intuïtie is volgens Kant kennis georganiseerd in ruimte en tijd. Dit suggereert dat een intuïtie een kennisitem is van een heel specifiek formaat. We concentreren ons op intuïties voor zover zij uit ruimtelijke relaties bestaan (temporele relaties spelen een rol in onder andere continuïteitsinferenties; we laten deze buiten beschouwing). In het licht hiervan stellen we voor dat een intuïtie een diagrammatisch kennisitem is. Volgens Kant betekent "het begrip van een driehoek construeren" aldus: het op diagrammatische wijze verbeelden van het begrip van een driehoek.

We beargumenteren dat volgens Kant een wiskundige (sommige van) de ruimtelijke relaties waaruit een intuïtie bestaat ook daadwerkelijk gebruikt tijdens een bewijsvoering. Aldus gebruikt, volgens Kant, de wiskundige intuïties wanneer hij een theorema's bewijst. Dit levert een antwoord op de hierboven gestelde vraag.

Een nevenresultaat van onze specifieke vraagstelling is dat Kants intuïtiebegrip in een interessant nieuw licht komt te staan. Onze benadering suggereert dat deze notie ongeveer als volgt getypeerd kan worden: een intuïtie constitueert een specifieke, dat wil zeggen diagrammatische, wijze van cognitieve organisatie. Bovendien blijkt volgens onze benaderingswijze wiskundig redeneren in Kants visie een vorm van diagrammatisch redeneren te zijn—inderdaad, het is zelfs *noodzakelijk* een vorm van diagrammatisch redeneren.

Waarom is het interessant om Kant de bovenstaande vraag voor te leggen?

Ten eerste zijn er historische redenen. Volgens Hintikka (in navolging van Beth) bestaat er geen diepe tegenstrijdigheid tussen Kants bewijsconceptie enerzijds en een op de moderne logica georiënteerde bewijsconceptie anderzijds. Sterker nog: Hintikka beweert dat Kants bewijsopvatting goed gereconstrueerd kan worden in termen van moderne logische systemen. In het licht hiervan biedt Hintikka ruimte voor een herwaardering van Kants ideeën.

Echter, volgens de moderne logica bewijst een wiskundige zijn theorema's door te redeneren middels het gebruik van proposities. Een dergelijke vorm van redeneren noemen we *propositioneel redeneren*. Propositioneel redeneren is sterk gerelateerd aan taal: een propositie kan uitgedrukt worden middels een beweerzin in een bepaalde taal. Een gevolg hiervan is dat volgens een op de logica georiënteerde bewijsopvatting een wiskundig bewijs volledig uitdrukbaar is in termen van beweerzinnen in een taal. Kortom, Hintikka's interpretatie van Kant doet geen goed recht aan Kants specifieke opvattingen omtrent de specifieke kennisitems waarmee een wiskundige redeneert.

Ten tweede zijn er ook systematische redenen die het zinnig maken Kants opvattingen nader te beschouwen. Zo geeft Friedman op basis van de moderne logica een radicaal tegenovergestelde waardering van Kant. In tegenstelling tot wat Hintikka denkt, zijn volgens Friedman Kants opvattingen omtrent wiskundige bewijsvoering min of meer obsoleet gemaakt door ontwikkelingen in de moderne logica. Wij denken dat Friedmans negatieve waardering van Kant enigszins overdreven is. In het bijzonder bestaat er vandaag de dag een groeiende belangstelling voor diagrammatische vormen van redeneren, onder andere binnen de cognitieve psychologie en de kunstmatige intelligentie (AI). Kants visie omtrent wiskundige bewijsvoering kan van dienst zijn teneinde de meer filosofische achtergronden van deze discussie in kaart te brengen.

Wij hebben de sterke neiging Kant te zien als iemand die diepe en buitengewoon waardevolle inzichten heeft in de meer cognitieve en methodologische aspecten van wiskundige bewijsvoering. Kants punt dat een wiskundige essentieel met intuïties redeneert, staat niet los hiervan. In het licht hiervan is het onterecht Kants visie te verwerpen omdat ze niet in overeenstemming zou zijn met hedendaagse op de logica georiënteerde bewijsconcepties (zoals bijvoorbeeld Friedman doet). Integendeel, Kants opvattingen omtrent wiskundige bewijzen dienen vanuit een breder perspectief beschouwd te worden. Voor Kant is wiskundig bewijzen niet een kwestie van logica alleen; een studie van wiskundige bewijzen dient zeker ook zaken van cognitieve en specifiek wiskundig-methodologische aard in zich op te nemen. Vanuit een modern gezichtspunt kan gezegd worden dat voor Kant logica, psychologie en wetenschapsfilosofie nauw op elkaar betrokken zijn. Veel nauwer dan vandaag de dag vaak het geval is.

Naast het inleidende hoofdstuk 1 is dit proefschrift voor de rest als volgt opgebouwd.

In het tweede hoofdstuk wordt een op de (moderne) logica georiënteerde bewijsconceptie aan een nader onderzoek onderworpen. De reden hierachter is dat een dergelijke bewijsconceptie wijdverbreid is in filosofische kringen en bovendien een belangrijke maatstaf vormt om Kants bewijsopvatting te interpreteren en te evalueren. Onze interesse gaat uit naar logica zoals opgevat in de traditie van Frege en het latere logisch empirisme. Binnen deze traditie wordt logica sterk gerelateerd aan wetenschappelijke methodologie. Voor wat betreft de logica, richten we ons op systemen van natuurlijke deductie. De reden is dat dergelijke systemen geacht worden de wiskundige redeneerwijze vrij adequaat te weerspiegelen. We beargumenteren dat de idee van redeneren als propositioneel redeneren nauw samenhangt met de aard van een van de specifieke instrumenten die door logici gebruikt wordt teneinde redeneringen te bestuderen. De instrumenten die we op het oog hebben zijn *talen* (logici gebruiken vooral artificiële talen).

We wijden in hoofdstuk 2 ook een kritische beschouwing aan een wetenschapsfilosofisch onderscheid dat voor een belangrijk deel de op logica georiënteerde bewijsconceptie lijkt te motiveren: het onderscheid tussen respectievelijk *context of discovery* en *context of justification*. We beargumenteren dat het onderscheid tussen deze twee contexten minder scherp is dan vaak wordt gedacht. Bovendien lijkt ook Kant van mening te zijn dat elementen betreffende de ontdekking van kennis en elementen betreffende de rechtvaardiging ervan soms nauw aan elkaar verweven te zijn.

In het derde hoofdstuk bespreken we de twee belangrijkste elementen van Kants filosofie van de wiskunde: respectievelijk zijn intuïtie- en constructiebegrip. Volgens Kant is een intuïtie een kennisitem dat individueel en onmiddellijk is. Kant onderscheidt verschillende soorten van intuïties. Het onderscheid tussen empirische intuïties en intuïties a priori is het meest belangrijk. Het zijn intuïties a priori die een rol spelen de (zuivere) wiskunde.

Kant maakt een onderscheid tussen de vorm en de inhoud van een intuïtie. De vorm van een intuïtie bestaat uit een aantal ruimtelijke (en temporele) relaties. De inhoud van een intuïtie kan gezien worden als de collectie van ongestructureerde relata (of *verschijningen*, zoals Kant zou zeggen) van deze relaties. De vraag of een intuïtie empirisch of a priori is komt volgens Kant neer op de vraag: wat is de bron van de inhoud van een intuïtie? Wanneer de inhoud van een intuïtie uit de zintuiglijke ervaring komt, dan hebben we te maken met een intuïtie a posteriori. Opmerkelijk genoeg komt volgens Kant de inhoud van een intuïtie a priori in de wiskunde voort uit de productieve verbeelding. Vele Kant commentatoren hebben het belang van de verbeelding voor Kants filosofie van de wiskunde over het hoofd gezien. Hierdoor heeft bijvoorbeeld Charles Parsons de sterke neiging Kants intuïtiebegrip, voor zoverre dat functioneert binnen diens filosofie van de wiskunde, vooral op te vatten in termen van empirische intuïties. We betogen tenslotte dat volgens Kant een intuïtie een diagrammatisch kennisitem is.

In het vierde hoofdstuk geven we een diepgravende analyse van een passage uit Kants *Kritik der reinen Vernunft*. In deze passage beschrijft Kant vrij nauwgezet hoe hij denkt dat een wiskundige een theorema uit de elementaire geometrie bewijst. Kant geeft ons de indruk dat in zijn opvatting een wiskundig bewijs getypeerd kan worden als een soort mentale animatie: een diagrammatisch kennisitem word geconstrueerd (bijvoorbeeld een driehoek). Vervolgens wordt dit kennisitem een aantal malen gemodificeerd (bijvoorbeeld door het toevoegen van hulplijnen). De creatie en modificatie van het diagrammatische kennisitem vormt bovendien het middel om de verschillende inferenties uit te voeren.

Een methodologisch raamwerk voor bewijzen in de wiskunde dat reeds door de neoplatoonse filosoof Proclus is besproken, dient als kader voor onze analyse. We betogen vervolgens in het bijzonder dat de inferenties die volgens Kant door een wiskundige uitgevoerd worden het beste als *constructief-inferentiële procedures* gekenmerkt kunnen worden. Volgens Kant is het een essentieel onderdeel van een inferentiële procedure dat er eerst een constructieve procedure (in de zin van Kant) uitgevoerd wordt.

In het vijfde hoofdstuk plaatsen we onze bevindingen in een wat breder kader. In het bijzonder willen we hier Kants these dat alle proposities van de wiskunde synthetisch a priori zijn vanuit ons cognitieve gezichtspunt opnieuw belichten. We bespreken ook Kants visie op de relatie tussen de wiskundige bewijsmethode en de (algemene) logica.

Volgens Kant bestaat er een belangrijk cognitief verschil tussen analytische en synthetische proposities. Analytische proposities zijn louter het product van een proces van conceptuele verheldering. In contrast daarmee vormen synthetische propostities het product van een proces tot uitbreiding van wetenschappelijke kennis. Nauw gerelateerd hieraan ligt de methodologische basis van analytische proposities in de (algemene) logica, en het *principium non contradictionis* in het bijzonder. De methodologische basis van synthetische proposities a priori ligt volgens Kant in de zogenaamde transcendentale logica. Voor wat betreft synthetische proposities a priori in de wiskunde in het bijzonder ligt de methodologische basis in de mogelijkheid van constructie van begrippen in termen van intuïties.

De syntheticiteit van een wiskundig theorema kan volgens Kant niet los gezien worden van de specifiek wiskundige methode die gebruikt wordt teneinde dit theorema te bewijzen. Deze methode zou volgens Kant een *speciale logica van de wiskunde* genoemd kunnen worden. De speciale logica van de wiskunde omvat de regels teneinde te denken over een specifiek wiskundig onderwerp. Deze speciale logica dient volgens Kant onderscheiden te worden van de algemene logica, welke volgens hem niet kan volstaan om op *wiskundige wijze* wiskundige theorema's te bewijzen. We betogen dat de speciale logica Proclus' methodologische raamwerk voor bewijzen in de wiskunde omvat.